NOTES ON ADDITIVELY DIVISIBLE COMMUTATIVE SEMIRINGS

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Abstract. Commutative semirings with divisible additive semigroup are studied. We show that a bounded finitely generated commutative additively divisible semiring is additively idempotent. One-generated commutative additively divisible semirings are treated in more detail.

It is well known that a commutative field is finite provided that it is a finitely generated ring. Consequently, no finitely generated commutative ring (whether unitary or not) contains a copy of the field $\mathbb{Q}$ of rational numbers. On the other hand, it seems to be an open problem whether a finitely generated (commutative) semiring S can contain a copy of the semiring (parasemifield) $\mathbb{Q}^+$ of positive rationals. Anyway, if S were such a (unitary) semiring with $1_S = 1_{\mathbb{Q}^+}$, then the additive semigroup $S(+)$ should be divisible. So far, all known examples of finitely generated additively divisible commutative semirings are additively idempotent. Hence a natural question arises, whether a finitely generated (commutative) semiring with divisible additive part has to be additively idempotent.

Analogous questions were studied for semigroups. According to [2, 2.5(iii)], there is a finitely generated non-commutative semigroup with a divisible element that is not idempotent. In this context it is of interest to ask whether there are non-finite but finitely generated divisible semigroups. This is in fact not true. Moreover, there exist infinite but finitely generated divisible groups (see [3] and [7]).

The present short note initiates a study of (finitely generated) additively divisible commutative semirings. In particular, we show that a finitely generated commutative additively divisible semiring that is bounded has to be additively idempotent. The one-generated case is treated in more detail, but remains an open problem.

1. Preliminaries

Throughout the paper, all algebraic structures involved (as semigroups, semirings, groups and rings) are assumed to be commutative, but, possibly, without additively and/or multiplicatively neutral elements. Consequently, a *semiring* is a non-empty set equipped with two commutative and associative binary operations, an addition and a multiplication, such that the multiplication distributes over the addition. A semiring S is called a ring if the additive semigroup of S is a group and S is called a *parasemifield* if the multiplicative semigroup of S is a non-trivial group.

We will use the usual notation: $\mathbb{N}$ for the semiring of positive integers, $\mathbb{N}_0$ for the semiring of non-negative integers, $\mathbb{Q}^+$ for the parasemifield of positive rationals and $\mathbb{Q}$ for the field of rationals.

2000 *Mathematics Subject Classification.* 16Y60.

Key words and phrases. commutative semiring, divisible semigroup.

This work is a part of the research project MSM00210839 financed by MŠMT. The first author was supported by the Grant Agency of Czech Republic, #201/09/0296 and the second author by the project LC 505 of Eduard Čech's Center for Algebra and Geometry.

Define a relation $\rho(k,t)$ on $\mathbb{N}$ for all $k,t \in \mathbb{N}$ by $(m,n) \in \rho(k,t)$ if and only if $m-n \in \mathbb{Z}t$ and either $m=n$ or $m \geq k$ and $n \geq k$. The following result is well known and quite easy to show:

Proposition 1.1. *The relations* $\mathrm{id}_{\mathbb{N}}(=\rho(k,0))$ *and* $\rho(k,t)$, $k,t \in \mathbb{N}$, *are just all congruences of the additive semigroup* $\mathbb{N}(+)$ *(and of the semiring* $\mathbb{N}$ *as well).*

Let S be a semiring. An element $a \in S$ is said to be *of finite order* if the cyclic subsemigroup $\mathbb{N}a = \{ka|\ k \in \mathbb{N}\}$, generated by the element a, is finite. We put $\mathrm{ord}(a) = \mathrm{card}(\mathbb{N}a)$ in this case. If $\mathcal{T}(S)$ denotes the set of elements of finite order, than either $\mathcal{T}(S) = \emptyset$ or $\mathcal{T}(S)$ is a *torsion* subsemiring (even an ideal) of S (i.e., every element of $\mathcal{T}(S)$ has finite order).

Lemma 1.2. *Let A be a non-empty subset of a semiring such that there exists $m \in \mathbb{N}$ with $\mathrm{ord}(a) \leq m$ for every $a \in A$. Then there exists $n \in \mathbb{N}$ such that $\mathrm{ord}(b) \leq n$ for every $b \in \langle A\rangle$. Moreover, there is $r \in \mathbb{N}$ such that $2rb = rb$ for every $b \in \langle A\rangle$.*

Proof. We have $\mathbb{N}a \cong \mathbb{N}(+)/\rho(k_a,t_a)$, where $k_a, t_a \in \mathbb{N}$, $k_a + t_a \leq m+1$. Furthermore, $2m_a a = m_a a$ for some $m_a \in \mathbb{N}$, $m_a \leq m+1$. Setting $r = (m+1)!$, we get $2rb = rb$ for every $b \in \langle A\rangle$. Of course, $\mathbb{N}b \cong \mathbb{N}(+)/\rho(k_b,t_b)$ and $\mathrm{ord}(b) = k_b + t_b - 1$. Since $2rb = rb$, we have $r \geq k_b$ and t_b divides $2r - r = r$. Consequently, $k_b + t_b - 1 \leq 2r - 1 = n$. □

Lemma 1.3. *Let $a,b \in S$ be such that $ka = la + b$ for some $k,l \in \mathbb{N}$, $k \neq l$. If $\mathrm{ord}(b)$ is finite, then $\mathrm{ord}(a)$ is so.*

Proof. There are $m,n \in \mathbb{N}$ such that $m < n$ and $mb = nb$. Then $nka = nla + nb = nla + mb = (n-m)la + m(la+b) = (n-m)la + mka = ((n-m)l + mk)a$. Since $k \neq l$, we see that $(n-m)k \neq (n-m)l$ and $nk \neq (n-m)l + mk$. Consequently, $\mathrm{ord}(a)$ is finite. □

Define a relation σ_S on a semiring S by $(a,b) \in \sigma_S$ if and only if $ma = mb$ for some $m \in \mathbb{N}$.

Define a relation τ_S on a semiring S by $(a,b) \in \tau_S$ if and only if $ma = nb$ for some $m,n \in \mathbb{N}$.

Lemma 1.4. *σ_S and τ_S are both congruences of S, $\sigma_S \subseteq \tau_S$, $\sigma_{S/\sigma_S} = id$ and $\tau_{S/\tau_S} = id$.*

Proof. It is easy. □

Lemma 1.5. *A semiring S is torsion, provided that the factor-semiring S/σ_S is torsion.*

Proof. For every $a \in S$ there are $k,l \in \mathbb{N}$ such that $(ka,la) \in \sigma_S$, $k < l$. Furthermore, there is $m \in \mathbb{N}$ with $mka = mla$. Clearly, $mk < ml$, and hence $\mathrm{ord}(a)$ is finite. □

Let S be a semiring and $o \notin S$ be a new element. Putting $x + o = o + x = x$ and $oo = xo = ox = o$ we get again a semiring $S \cup \{o\}$. Consider now the semiring $T = \mathbb{N}_0 \times (S \cup \{o\})$ equipped with component-wise addition and multiplication given as $(n,a)(m,b) = (nm, ma + nb + ab)$ for every $n,m \in \mathbb{N}_0$ and $a,b \in S \cup \{o\}$. Denote $\mathcal{U}(S) = T \setminus \{(0,o)\}$ the (unitary) subsemiring.

Now, S can always be treated as a natural unitary $\mathcal{U}(S)$-semimodule, with $(n,a)x = nx + ax$ for every $(n,a) \in \mathcal{U}(S)$, $x \in S$. (Here $(n,o)x = nx$ and $(0,a)x = ax$ for $n \in \mathbb{N}$ and $a,x \in S$.)

Lemma 1.6. *Let S be a semiring. If $w \in S$, $a,b,c \in \mathcal{U}(S)w$ and $m \in \mathbb{N}$ are such that $ma = mb$ and $mc = w$, then $a = b$.*

Proof. For every $d \in \mathcal{U}(S)w$, there is $\alpha_d \in \mathcal{U}(S)$ with $d = \alpha_d w$. Now, $a = \alpha_a w = \alpha_a mc = \alpha_c \alpha_a mw = \alpha_c ma = \alpha_c mb = \alpha_c \alpha_b mw = \alpha_b mc = \alpha_b w = b$. □

Lemma 1.7. *Let T be a subsemiring of a semiring S and let $\mathcal{D}_S(T) = \{a \in S \mid \mathbb{N}a \cap T \neq \emptyset\}$. Then:*

(i) *$\mathcal{D}_S(T)$ is a subsemiring of S and $T \subseteq \mathcal{D}_S(T)$.*
(ii) *$\mathcal{D}_S(\mathcal{D}_S(T)) = \mathcal{D}_S(T)$.*

Proof. It is easy. □

2. Additively divisible semirings

Let $S(= S(+))$ be a semigroup. An element $a \in S$ is called *divisible* (*uniquely divisible*, resp.) if for every $n \in \mathbb{N}$ there exists $b \in T$ (a unique, resp.) such that $a = nb$.

A semigroup S is called *divisible* (*uniquely divisible*, resp.) if every element of S is divisible (uniquely divisible, resp.). Clearly, S is divisible iff $S = nS$ for every $n \in \mathbb{N}$. The class of divisible semigroups is closed under taking homomorphic images and cartesian products and contains all divisible groups and all semilattices (i.e., idempotent semigroups).

A semiring is called *additively divisible* (*additively uniquely divisible*, resp.) if its additive part is a divisible semigroup (uniquely divisible semigroup, resp.). The semiring $\mathbb{Q}^+$ is additively (uniquely) divisible.

Note, that any semigroup $S(+)$ with an idempotent element e (i.e. $e + e = e$) can always be treated as a semiring with a constant multiplication given as $ab = e$ for every $a, b \in S$.

The following theorem is a consequence of [2, 2.14(i)]. But for completeness (and using stronger assumptions) we provide a simpler proof.

Theorem 2.1. ([2]) *A semigroup S is finitely generated and divisible if and only if S is a finite semilattice.*

Proof. Only the direct implication needs a proof. Let S be a divisible semigroup generated by a finite set A. We will proceed by induction on the number m of non-idempotent generators.

If $m = 0$, then S is generated by a set of idempotents and it follows easily that S is a finite semilattice. Now, assume that $m \geq 1$ and choose $a \in A$ with $2a \neq a$. Define a relation ρ_a on S by $(u, v) \in \rho_a$ if and only if $u + ka = v + la$ for some $k, l \in \mathbb{N}$. Then ρ_a is a congruence of the semigroup and S/ρ_a is a finite semilattice by induction. Since S is divisible, we have $a = 2b$ for some $b \in S$ and $(a, b) = (2b, b) \in \rho_a$ (since S/ρ_a is idempotent). Then $ka = b + la$ for suitable $k, l \in \mathbb{N}$ and we get $2ka = 2b + 2la = (2l + 1)a$. Since $2k \neq 2l + 1$, we conclude that the cyclic subsemigroup $\mathbb{N}a$ generated by $\{a\}$ is finite.

We have proved that $\mathbb{N}a$ is finite for every $a \in A$. Since S is generated by A and A is finite, one checks easily that S is finite, too. In particular, for every $u \in S$ there is $n_u \in \mathbb{N}$ such that $2n_u u = n_u u$. If $n = \prod n_u$, then $2nu = nu$. Since S is divisible, $nS = S$ and it follows that S is a finite semilattice. □

Remark 2.2. Let S be a divisible semiring. Then both S/σ_S and S/τ_S are additively uniquely divisible semirings.

Theorem 2.3. *A torsion divisible semiring S is additively idempotent if at least one of the following two condition is satisfied:*

(i) *S is bounded (i.e., there exists $m \in \mathbb{N}$ with $\operatorname{ord}(a) \leq m$ for every $a \in S$.);*
(ii) *S is additively uniquely divisible.*

Proof. (i) If S is bounded, then by 1.2 there exists $n \in \mathbb{N}$ such that $2na = na$ for every $a \in S$. Since S is divisible, we have $a = nb$, and so $2a = 2nb = nb = a$.

(ii) If S is additively uniquely divisible and $a \in S$, then $2ka = ka$ for some $k \in \mathbb{N}$, so that $k(2a) = ka$ and $2a = a$. □

Corollary 2.4. *If S is a torsion divisible semiring, then σ_S is just the smallest congruence of S such that the corresponding factor-semiring is additively idempotent.*

Corollary 2.5. *Let S be an additively uniquely divisible semiring. Then $\mathcal{T}(S)$ is additively idempotent.*

Example 2.6. Consider a non-trivial semilattice S and the quasicyclic p-group $\mathbb{Z}_{p^\infty}$. Then the product $S \times \mathbb{Z}_{p^\infty}$ is a torsion divisible semigroup that is neither a semilattice nor a group.

3. Additively divisible semirings - continued

Throughout this section, let S be an additively divisible semiring.

Proposition 3.1. (i) *Both the factor-semirings S/σ_S and S/τ_S are additively uniquely divisible.*

(ii) *If $\mathcal{T}(S) \neq \emptyset$, then $\mathcal{T}(S)$ is an additively divisible ideal S.*

Proof. It is easy. □

Proposition 3.2. *Assume that the semiring S is generated as an S-semimodule by a subset A such that $\operatorname{ord}(a) \leq m$ for some $m \in \mathbb{N}$ and all $a \in A$. Then S is additively idempotent.*

Proof. Put $B = \{b \in S | \operatorname{ord}(b) \leq m\}$. Then $A \subseteq B$ and $sb \in B$ for all $s \in S$ and $b \in B$. Consequently, $\langle B \rangle = S$ and, by 1.2, there is $n \in \mathbb{N}$ with $\operatorname{ord}(x) \leq n$ for every $x \in S$. Now, it remains to use 2.3(i). □

Corollary 3.3. *The semiring S is additively idempotent, provided that it is generated as an S-semimodule by a finite set of elements of finite orders.*

Corollary 3.4. *The semiring S is additively idempotent, provided that it is torsion and finitely generated.*

Remark 3.5. (i) The zero multiplication ring defined on $\mathbb{Z}_{p^\infty}$ is both additively divisible and additively torsion. Of course, the ring is neither additively idempotent nor finitely generated. The (semi)group $\mathbb{Z}_{p^\infty}(+)$ is not uniquely divisible.

(ii) Let R be a (non-zero) finitely generated ring (not necessary with unit). Then R has at least one maximal ideal I and the factor-ring R/I is a finitely generated simple ring. However, any such a ring is finite and consequently, R is not additively divisible.

Remark 3.6. Assume that S is additively uniquely divisible.

(i) For all $m, n \in \mathbb{N}$ and $a \in S$, there is a uniquely determined $b \in S$ such that $ma = nb$ and we put $(m/n)a = b$. If $m_1, n_1 \in \mathbb{N}$ and $b_1 \in S$ are such that $m/n = m_1/n_1$ and $m_1 a = n_1 b_1$, then $k = mn_1 = m_1 n$ and $kb_1 = mm_1 a = kb$ and $b_1 = b$. Consequently, we get a (scalar) multiplication $\mathbb{Q}^+ \times S \to S$ (one checks easily that $q(a_1 + a_2) = qa_1 + qa_2$, $(q_1 + q_2)a = q_1 a + q_2 a$, $q_1(q_2 a) = (q_1 q_2)a$ and $1a = a$ for all $q_1, q_2 \in \mathbb{Q}^+$ and $a_1, a_2, a \in S$) and S becomes a unitary $\mathbb{Q}^+$-semimodule. Furthermore, $qa_1 \cdot a_2 = a_1 \cdot qa_2$ for all $q \in \mathbb{Q}^+$ and $a_1, a_2 \in S$, and therefore S is a unitary $\mathbb{Q}^+$-algebra.

(ii) Let $a \in S$ be multiplicatively but not additively idempotent (i.e., $a^2 = a \neq 2a$). Put $Q = \mathbb{Q}^+ a$. Then Q is a subalgebra of the $\mathbb{Q}^+$-algebra S and the mapping

$\varphi : q \mapsto qa$ is a homomorphism of the $\mathbb{Q}^+$-algebras and, of course, of the semirings as well. Since $a \neq 2a$, we have $\ker(\varphi) \neq \mathbb{Q}^+ \times \mathbb{Q}^+$. But $\mathbb{Q}^+$ is a congruence-simple semiring and it follows that $\ker(\varphi) = id$. Consequently, $Q \cong \mathbb{Q}^+$.

Put $T = Sa$. Then T is an ideal of the $\mathbb{Q}^+$-algebra S, $Q \subseteq T$ (we have $qa = qa \cdot a \in T$) and $a = 1_Q = 1_T$ is a multiplicatively neutral element of T. The mapping $s \mapsto sa$, $s \in S$, is a homomorphism of the $\mathbb{Q}^+$-algebras. Consequently, T is additively uniquely divisible. Furthermore, T is a finitely generated semiring, provided that S is so.

Proposition 3.7. *Assume that $1_S \in S$. Then:*

(i) *S is additively uniquely divisible.*
(ii) *Either S is additively idempotent or S contains a subsemiring Q such that $Q \cong \mathbb{Q}^+$ and $1_S = 1_Q$.*
(iii) *If $\mathrm{ord}(1_S)$ is finite, then S is additively idempotent.*

Proof. For every $m \in \mathbb{N}$, there is $s_m \in S$ such that $1_S = ms_m$. That is, $s_m = (m1_S)^{-1}$. If $ma = mb$, then $a = s_m ma = s_m mb = b$ and we see that S is additively uniquely divisible. The rest is clear from 3.6. □

Proposition 3.8. *If the semiring S is non-trivial and additively cancellative, then S is not finitely generated.*

Proof. The difference ring $R = S - S$ of S is additively divisible, and hence it is not finitely generated by 3.5(ii). Then S is not finitely generated either. □

Lemma 3.9. *If T is a subsemiring of S, then the subsemiring $\mathcal{D}_S(T)$ (see 1.7) is additively divisible.*

Proof. It is easy. □

4. One-generated additively divisible semirings

In this section, let S be an additively divisible semiring generated by a single element $w \in S$.

Proposition 4.1. *The semiring S is additively uniquely divisible.*

Proof. Follows from 1.6. □

Lemma 4.2. *The semiring S is additively idempotent, provided that $\mathrm{ord}(w^m)$ is finite for some $m \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ be the smallest number with $\mathrm{ord}(w^n)$ finite. If $n = 1$, then the result follows from 3.3, and so we assume, for contrary, that $n \geq 2$. Since $S(+)$ is divisible, there are $v \in S$, $k \in \mathbb{N}_0$ and a polynomial $f(x) \in \mathbb{N}_0[x] \cdot x$ such that $w = 2v$ and $v = kw + w^2 f(w)$. Hence $w^{n-1} = 2kw^{n-1} + 2w^n f(w)$. By assumption, $2w^n f(w)$ is of finite order. If $k = 0$, then clearly $\mathrm{ord}(w^{n-1})$ is finite, and if $k \geq 1$, then $\mathrm{ord}(w^{n-1})$ is finite, by 1.3, the final contradiction. □

For $n \in \mathbb{N}$ denote $S^n = \langle \{a_1 \dots a_n |\ a_i \in S\} \rangle$.

Lemma 4.3. *If $u \in S$ is such that $w = wu$, then $u = 1_S$.*

Proof. Let $w = wu$. Since $S = \mathcal{U}(S)w$, for every $a \in S$ there is $\alpha_a \in \mathcal{U}(S)$ such that $a = \alpha_a w$. Hence $a = \alpha_a w = \alpha_a wu = au$ for every $a \in S$. Thus $u = 1_S$. □

Corollary 4.4. *$1_S \in S$ (i.e. S is unitary) if and only if $S^2 = S$. In this case:*

(i) *$w^{-1} \in S$.*
(ii) *$S^n = S^m$ for all $n, m \in \mathbb{N}$.*

Proof. Let $S^2 = S$. Then $w \in S = S^2$ and there is a non-zero polynomial $f(x) \in \mathbb{N}_0[x] \cdot x$ such that $w = wf(w)$. By 4.3, $f(w) = 1_S$. Now, since $1_S \in S = S^2$, we have similarly that $1_S = wu$ for some $u \in S$. Hence $w^{-1} = u \in S$. The rest is obvious. □

Lemma 4.5.

(i) *For every $n \in \mathbb{N}$ there are $a, b \in S$ such that $w + aw^n = 2w + bw^n$.*
(ii) *For all $n, m \in \mathbb{N}$ there are $c, d \in S$ such that $mw^n + mcw^n = w^n + mdw^n$.*

Proof. Let $n \in \mathbb{N}$. Consider a congruence ρ of the semiring S such that $(x, y) \in \rho$ iff $x + a'w^n = y + b'w^n$ for some $a', b' \in S$. Then the additive semigroup S/ρ is finitely generated, hence idempotent by 2.1. Thus $(w, 2w) \in \rho$ and there are $a, b \in S$ such that $w + aw^n = 2w + bw^n$.

Now, let $m \in \mathbb{N}$. Since S is additively divisibile, there is $u \in S$ such that $w^n = mu$. By the additive idempotency of S/ρ we have $(w^n, u) = (mu, u) \in \rho$. Hence there are $c, d \in S$ such that $w^n + cw^n = u + dw^n$. Finally, $mw^n + mcw^n = w^n + mdw^n$. □

Lemma 4.6. *Let $k \in \mathbb{N}$, $k \geq 2$ and $u \in S \cup \{0\}$ be such that $w = kw + u$. Then there exists $a \in S \cup \{0\}$ such that $x = 2x + ax$ for every $x \in S$.*

Proof. Let $u = mw + wf(w)$, where $m \in \mathbb{N}_0$ and $f(x) \in \mathbb{N}_0[x] \cdot x$. If $f(x) = 0$, then $\mathrm{ord}(w)$ is finite, S is idempotent by 3.3 and we can put $a = 0$.

Hence assume that $f(x) \neq 0$. Put $n = m + k$ and $b = f(w) \in S$. We have $w = nw + wb$. Adding $(n-2)w$ to both sides of this equality, we get $(n-1)w = 2(n-1)w + wb$. Since w is a generator and S is additively divisible we have $b = (n-1)a$ for some $a \in S$ and for every $x \in S$ there is $\alpha_x \in \mathcal{U}(S)$ such that $x = (n-1)\alpha_x w$. Hence $x = \alpha_x(n-1)w = \alpha_x(2(n-1)w + (n-1)wa) = 2x + xa$ for every $x \in S$. □

Remark 4.7. Let $x, v \in S$ be such that $x = 2x + v$. Put $u = x + v$. Then:

(i) u is an idempotent and $x = x + u$.
(ii) Let $v' \in S$ be such that $x = 2x + v'$. Then $u = x + v'$.
(iii) The set $\{a \in S |\ x = x + a\}$ is a subsemigroup of $S(+)$.

Lemma 4.8. *S is additively divisible if and only if for every prime integer p there is $v_p \in S$ with $w = pv_p$.*

Proof. The direct implication is trivial. Conversely, if $w = pv_p$, then $w \in pS$, and so $pS = S$, since pS is an ideal of the semiring S. Furthermore, given $m \in \mathbb{N}$, $m \geq 2$, we have $m = p_1^{k_1} \dots p_n^{k_n}$, and hence $mS = S$ as well. □

Remark 4.9. (i) Let $p_1, \dots, p_n$ be prime numbers. Put $m = p_1 \cdots p_n$ and $S = \mathbb{Z}_{m-1}$. Then S is generated (as a semiring) by the element $w = [1]_{m-1}$ and $w = mw$. Hence for every $i = 1, \dots, n$ there is v_{p_i} such that $w = p_i v_{p_i}$, but S is not additively idempotent.

(ii) Let $S = \mathbb{Z}_2$. Then S is one-generated and for every prime number $p \neq 2$ and every $x \in S$ we have $x = px$, but S is again not additively idempotent.

Clearly, S is additively divisible, iff for every $n \in \mathbb{N}$ there is a non-zero polynomial $f_n(x) \in \mathbb{N}_0[x] \cdot x$ such that $w = nf_n(w)$. By [2, 2.5(i)], every divisible element in a finitely generated commutative semigroup has to be idempotent. Assuming now, that the degree of the polynomials f_n is bounded, we get that S is additively idempotent. To illustrate some other situations and techniques see the next examples.

Example 4.10. Let $n, m, k, l \in \mathbb{N}$ be such that $n^{l-1} \neq m^{k-1}$ and suppose that $w = nw^k$ and $w = mw^l$. Then S is additively idempotent.

Indeed, $nm^k w^{kl} = n(mw^l)^k = nw^k = w = mw^l = m(nw^k)^l = mn^l w^{kl}$. Since $nm^k \neq mn^l$, w^{kl} is of finite order and S is additively idempotent by 4.2.

Example 4.11. Let $w = 2(w + w^2)$ and $w = 3(w + w^3)$. Then S is additively idempotent.

Indeed, adding these two equalities we get $2w = 5w + 2w^2 + 3w^3$. Now, since $w^2 = 2w^2 + 2w^3$ we can substitute $2w = 5w + (2w^2 + 2w^3) + w^3 = 5w + w^2 + w^3$. Hence $4w = 10w + (2w^2 + 2w^3)$ and by the same substitution we get $4w = 10w + w^2$. Finally, $8w = 20w + 2w^2 = 18w + (2w + 2w^2) = 19w$ and w is of finite order. Thus S is additively idempotent by 4.2.

Lemma 4.12. *$T_S = \{\varphi \in \mathrm{End}(S(+))|\ (\exists \alpha \in \mathcal{U}(S))(\forall x \in S)\ \varphi(x) = \alpha x\}$ is a unitary additively divisible two-generated semiring.*

Moreover, S is additively idempotent if and only if T_S is so.

Proof. We only need to show, that $\mathrm{id}_S \in T_S$ is an additively divisible element. Let $n \in \mathbb{N}$. Then $w = na$ for some $a \in S$ and $a = \alpha_a w$ for some $\alpha_a \in \mathcal{U}(S)$. Now, for every $x \in S$ there is $\alpha_x \in \mathcal{U}(S)$ such that $x = \alpha_x w$. Hence $x = \alpha_x w = \alpha_x n \alpha_a w = n\alpha_a x$. Thus $\mathrm{id}_S = n\alpha_a$. The rest is easy. $\square$

5. One-generated additively divisible semirings - continued

This section is an immediate continuation of the preceding one. We assume that S is non-trivial. Since the congruence $S \times S$ is generated by $\{(w, 2w), (w, w^2)\}$, S has at least one (proper) maximal congruence λ and the factor-semiring $T = S/\lambda$ is congruence-simple. Of course, T is additively divisible and one-generated. We denote by ψ the natural projection of S onto T. According to [1, 10.1], T is an additively idempotent semiring, T is finite and just one of the following three cases takes place:

(a) $T \cong Z_3$;
(b) $T \cong Z_4$;
(c) $T \cong V(G)$ for a non-trivial finite cyclic group G.

The definitions are as follows:

$$Z_3: \quad \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \qquad\qquad Z_4: \quad \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

Let $G(\cdot)$ be an abelian group, $o \notin G$. Put $V(G) = G \cup \{o\}$ and define $x + y = y + x = o$, $x + x = x$ and $xo = ox = o$ for all $x, y \in V(G), x \neq y$.

Theorem 5.1. *If $T \cong Z_3$, then S is additively idempotent.*

Proof. The congruence λ has just two blocks, say $A = \psi^{-1}(0)$ and $B = \psi^{-1}(1)$, where A is a bi-ideal of S, $SS \subseteq A$ and $w \in B$, $B + B \subseteq B$. Then $w^2, w^3, \dots \in A$ and it follows that $B = \mathbb{N}w$. On the other hand, $w = 2v$ for some $v \in S$ and we have $v \in B$. This means that $\mathrm{ord}(w)$ is finite and it follows from 4.2 that S is additively idempotent. $\square$

Proposition 5.2. *Assume that $T \cong Z_4$. Then:*

(i) *The congruence λ has just two blocks $A = \psi^{-1}(0)$ and $B = \psi^{-1}(1)$.*
(ii) *A is an ideal of $S(+)$ and $w \in A$.*

(iii) *B is an ideal of S and $SS \subseteq B$.*
(iv) *B is an additively uniquely divisible semiring and B is generated by the set $\{w^2, w^3\}$.*
(v) *$A(+)$ is a uniquely divisible semigroup.*
(vi) *$1_S \notin S$.*

Proof. We have $A = \{n_1w + n_2w^2 + \cdots + n_kw^k |\ k \in \mathbb{N}, n_i \in \mathbb{N}_0, n_1 \neq 0\}$ and $B = \{m_2w^2 + \cdots + m_lw^l |\ 2 \leq l \in \mathbb{N}, m_j \in \mathbb{N}_0, \sum m_j \neq 0\}$. The rest follows from 4.1, 4.3 and 4.4. □

In the remaining part of this section, assume that $T \cong V(\mathbb{Z}_m) = \mathbb{Z}_m \cup \{o\} = \{o, 0, 1, \ldots, m-1\}$ for some $2 \leq m \in \mathbb{N}$ (here $\mathbb{Z}_m(+) = \{0, 1, \ldots, m-1\}$ is the m-element cyclic group of integers modulo m). Furthermore, put $A = \psi^{-1}(o)$ and $B_k = \psi^{-1}(k)$ for every $k = 0, 1, \ldots, m-1$. Since $\psi(w)$ has to be a generator of $\mathbb{Z}_m$, we can (without loss of generality) assume that $w \in B_1$, $w^2 \in B_2, \ldots,$ $w^{m-1} \in B_{m-1}$ and $w^m \in B_0$.

Proposition 5.3.
(i) *$B_0 = \{n_1w^m + n_2w^{2m} + \cdots + n_lw^{lm} |\ l \in \mathbb{N}, n_i \in \mathbb{N}_0, \sum n_i \neq 0\}$.*
(ii) *B_0 is a subsemiring of S and B_0 is an additively uniquely divisible semiring generated by a single element, namely by the element w^m.*
(iii) *$B_1, \ldots, B_{m-1}$ are uniquely divisible subsemigroups of $S(+)$.*
(iv) *$B_k = \{n_1w^k + n_2w^{k+m} + \cdots + n_lw^{k+(l-1)m} |\ l \in \mathbb{N}, n_i \in \mathbb{N}_0, \sum n_i \neq 0\}$ for every $1 \leq k \leq m-1$.*
(v) *$B_kB_l \subseteq B_t$, $t = k + l \pmod m$ for all $0 \leq k, l \leq m-1$ and $B_k + B_l \subseteq A$ for $k \neq l$.*
(vi) *A is an bi-ideal of S and an additively uniquely divisible semiring.*

Proof. It is easy. □

Proposition 5.4. *Assume that $1_S \in S$. Then:*
(i) *$1_S \in B_0$ and $1_S = n_1w^m + n_2w^{2m} + \cdots + n_lw^{lm}$, where $l \in \mathbb{N}$, $n_i \in \mathbb{N}_0$ and $\sum n_i \neq 0$.*
(ii) *$w^{-1} = n_1w^{m-1} + \cdots + n_lw^{lm-1} \in B_{m-1}$.*
(iii) *$w^{-m} = n_11_S + n_2w^m + \cdots + n_lw^{(l-1)m} \in B_0$.*
(iv) *S is additively idempotent if and only if B_0 is so (i.e., iff $1_S = 2_S$).*
(v) *If S is not additively idempotent, then $w^t \neq 1_S$ for any $t \in \mathbb{N}$.*

Proof. The first four assertions follow easily from 5.3. To show (v), assume on the contrary that $w^t = 1_S$ for some $t \in \mathbb{N}$. Then the semigroup $S(+)$ is generated by $\{1_S, w, \ldots, w^t\}$. Hence S is additively idempotent by 2.1, a contradiction. □

In the end, note that although the one-generated structure is quite simple on the first look, it seems that it will involve a lot of effort to solve even this case.

6. A few conjectures

In this last section we present some other open questions that are influenced by our main problem (namely by the conjecture (A) - see below).

Example 6.1. Given a multiplicative abelian group G and an element $o \notin G$, put $U(G) = G \cup \{o\}$ and define addition and multiplication on $U(G)$ (extending the multiplication on G) by $x + y = xo = ox = o$ for all $x, y \in U(G)$. Then $U(G)$ becomes an ideal-simple semiring.

Consider the following statements:
(A) Every finitely generated additively divisible semiring is additively idempotent.

(A1) Every finitely generated additively uniquely divisible semiring is additively idempotent.

(B) No finitely generated semiring contains a copy of $\mathbb{Q}^+$.

(B1) No finitely generated semiring with a unit element contains a copy of $\mathbb{Q}^+$ sharing the unit.

(C) Every finitely generated infinite and ideal-simple semiring is additively idempotent or a copy of the semiring $U(G)$ (see 6.1) for an infinite finitely generated abelian group G.

(D) Every parasemifield that is finitely generated as a semiring is additively idempotent.

Proposition 6.2. (A) $\Leftrightarrow$ (A1) $\Rightarrow$ (B) $\Leftrightarrow$ (B1) $\Rightarrow$ (C) $\Leftrightarrow$ (D).

Proof. First, it is clear that (A) $\Rightarrow$ (A1) and (B) $\Rightarrow$ (B1). Furthermore, (C) $\Leftrightarrow$ (D) by [4, 5.1]. Now, assume that (A1) is true and let S be a finitely generated additively divisible semiring. By 3.1, S/σ_S is additively uniquely divisible and, of course, this semiring inherits the property of being finitely generated. By (A1), the semiring S/σ_S is additively idempotent, and hence the semiring S is additively torsion by 1.5. Finally, S is additively idempotent by 3.4. We have shown that (A1) $\Rightarrow$ (A) and consequently, (A) $\Leftrightarrow$ (A1).

Next, let (B1) be true and let S be a finitely generated semiring containing a subsemiring $Q \cong \mathbb{Q}^+$. Put $P = S1_Q$. Then P is an ideal of S, $1_Q = 1_P$, $Q \subseteq P$ and the map $s \mapsto s1_Q$ is a homomorphism of S onto P. Thus P is a finitely generated semiring and this is a contradiction with (B1). We have shown that (B1) $\Rightarrow$ (B) and consequently, (B) $\Leftrightarrow$ (B1).

Now, we are going to show that (A) $\Rightarrow$ (B1). Indeed, let S be a finitely generated semiring such that $1_S \in S$ and S contains a subsemiring Q with $1_S \in Q$ and $Q \cong \mathbb{Q}^+$. If $a \in S$ and $m \in \mathbb{N}$, then $b = (m1_S)^{-1}a \in S$ and $mb = a$. It follows that S is additively divisible, and hence additively idempotent by (A). But Q is not so, a contradiction. We have shown that (A) $\Rightarrow$ (B1).

It remains to show that (B1) $\Rightarrow$ (D). Let S be a parasemifield that is not additively idempotent and let Q denote the subparasemifield generated by 1_S. Then $Q \cong \mathbb{Q}^+$, $1_Q = 1_S$ and S is not finitely generated due to (B1). □

Note that using the Birkhoff's theorem we can consider an equivalent version of the conjecture (A):

(A') Every finitely generated subdirectly irreducible additively divisible semiring is additively idempotent.

Of course, it would be sufficient if such a semiring were finite. Unfortunately, this is not true. Assume for instance the semiring $S = V(G)$ with $G = \mathbb{Z}(+)$. By [1, 10.1], S is simple and two-generated. Nevertheless, it is an open question whether also one-generated subdirectly irreducible additively divisible semiring can be infinite.

Finally, Mal'cev [6] proved that every finitely generated commutative semigroup is residually finite (i.e. it is a subdirect product of finite semigroups). Notice, that also the additive part of a freely finitely generated additively idempotent semiring is a residually finite semigroup. If this is true also for every finitely generated additively divisible semiring, we get a nice positive answer to the conjecture (A).

References

[1] R. El Bashir, J. Hurt, A. Jančařík and T. Kepka, *Simple commutative semirings*, J. Algebra **236**(2001), 277-306.

[2] W. E. Clark, W. C. Holland and G. J. Székely, *Decompositions in discrete semigroups*, Studia Sc. Math. Hungarica, **34**(1998), 15-23.

[3] V. S. Guba, *A finitely generated complete group*, Izv. Academy Nauk, SSSR, Ser. Math. **50(5)**(1986), 883-924.

[4] V. Kala and T.Kepka, *A note on finitely generated ideal-simple commutative semirings*, Comment. Math. Univ. Carolinae **49**(2008), 1-9.

[5] V. Kala, T. Kepka and M. Korbelář, *Notes on commutative parasemifields* (preprint)

[6] A. I. Mal'cev, *On homomorphisms into finite groups*, Lecture Notes of Ivanovsk Pedagog. Inst. **18**(1958), 49-60.

[7] A. Ju. Olshanskii, Geometry of Dening Relations in Groups, Nauka, Moscow, (1989) (in Russian) (English translation: Kluwer, 1991).

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