CLASSES OF CATEGORIES ASSOCIATED TO SIMPLICIAL FUNCTORS

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Abstract. Given a simplicial model category $\mathbf{M}$ and a simplicial valued simplicial functor F defined on $\mathbf{M}$, we associate to the pair ($\mathbf{M}$,F) a class of small categories and we study its properties. Homotopy sifted categories provide an example of such class. We introduce and study homotopy sifted-flat functors.

Homotopy sifted categories were introduced in [11] as a homotopy theoretical analogue of sifted categories. In this note a class of small categories is associated to every simplicial valued simplicial functor defined on a simplicial model category. For suitable simplicial functors we make a formal study of this class of categories. Homotopy sifted categories is the motivating, and one of the simplest, example of such a class. We recover and complement the results from [11] related to this subject.

As a possible analogue of the notion of sifted flat functors of J. Adamek and J. Rosicky [1], we define and study homotopy sifted-flat functors.

Notations and conventions. We denote by **Cat** the category of small categories and by $\mathbf{S}$ the category of simplicial sets. We denote by N the nerve functor.

Let $\mathbf{M}$ be an arbitrary category. We denote by $(\mathbf{Cat} \downarrow \mathbf{M})$ the category with objects pairs $(I, \mathbf{X} : I \to \mathbf{M})$, where $I \in \mathbf{Cat}$, and arrows $(F, \alpha) : (I, \mathbf{X} : I \to \mathbf{M}) \to (J, \mathbf{Y} : J \to \mathbf{M})$ those pairs consisting of a functor $F : I \to J$ and a natural transformation $\alpha : \mathbf{X} \Rightarrow \mathbf{Y}F$.

Let $\mathbf{M}$ be a simplicial model category. If I is a small category and $\mathbf{X} : I \to \mathbf{M}$ is a functor, $hocolim_I \mathbf{X}$ stands for the homotopy colimit of $\mathbf{X}$, as defined in ([8], 18.1.2). One has that $hocolim$ is a functor $(\mathbf{Cat} \downarrow \mathbf{M}) \to \mathbf{M}$. This follows from the fact that for any functor $F : I \to J$, the natural map $hocolim_I \mathbf{X}F \to hocolim_J \mathbf{X}$ is natural in $\mathbf{X} : J \to \mathbf{M}$.

Let $\mathbf{M}$ be a cofibrantly generated model category. Given a small category I, the functor category $\mathbf{M}^I$ shall always be regarded as having the projective model structure, in which the fibrations and the weak equivalences are defined level-wise.

Let I be a small category and let $F : I \to Set$ be a functor. The category ElF of elements of F is defined in the following way. The objects of ElF are pairs (i, a), where $i \in ObI$ and $a \in F_i$. An arrow of ElF from (i, a) to (j, b) is an arrow $f : i \to j$ of I such that $F_f(a) = b$. The association $F \mapsto ElF$ is a functor $Set^I \to \mathbf{Cat}$.

Let I be a small category. Let $y^* : \Delta^{op} \times I \to \mathbf{S}^I$ be the contravariant Yoneda functor. For every object $\mathbf{X}$ of $\mathbf{S}^I$, we denote by $[I, \mathbf{X}]$ the composite $El\mathbf{X} \to \Delta^{op} \times I \xrightarrow{y^*} \mathbf{S}^I$.

1. Classes of categories associated to simplicial functors

Let $\mathbf{M}$ be a simplicial model category and $F : \mathbf{M} \to \mathbf{S}$ a simplicial functor.

Definition 1.1. *We let* $< \mathbf{M}, F >$ *be the full subcategory of* **Cat** *consisting of those objects* I *such that for all* $\mathbf{X} : I \to \mathbf{M}$ *taking cofibrant values, the canonical*

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map

$$hocolim_I F\mathbf{X} \to Fhocolim_I \mathbf{X}$$

is a weak homotopy equivalence.

It readily follows from the definition that
(a) $<\mathbf{M},F>$ has finite products; in particular $<\mathbf{M},F>$ contains the terminal category 1,
(b) a retract in **Cat** of an object of $<\mathbf{M},F>$ is in $<\mathbf{M},F>$, and
(c) if $\alpha : F \Rightarrow G : \mathbf{M} \to \mathbf{S}$ is a simplicial natural transformation which is a levelwise weak homotopy equivalence on cofibrant objects, then $<\mathbf{M},F>=<\mathbf{M},G>$.
Part (c) implies that $<\mathbf{M},F>=<\mathbf{M},Ex^{\infty}F>$ and $<\mathbf{S},Ex^{\infty}>=\mathbf{Cat}$, where Ex^{∞} is Kan's completion functor (see, for instance, ([7], III, 4)).

We define $<\mathbf{M},F>_f$ to be the full subcategory of **Cat** consisting of those objects I such that for all $\mathbf{X} : I \to \mathbf{M}$ taking cofibrant-fibrant values, the canonical map

$$hocolim_I F\mathbf{X} \to Fhocolim_I \mathbf{X}$$

is a weak homotopy equivalence. One has $<\mathbf{M},F>\subset<\mathbf{M},F>_f$. The converse holds if F preserves weak equivalences between cofibrant objects. We recall that F, as a simplicial functor, preserves simplicial homotopy equivalences, therefore it always preserves weak equivalences between cofibrant-fibrant objects ([8], 9.5.24(2)).

If $\mathbf{M}$ is moreover cofibrantly generated, we define $<\mathbf{M},F>_{cf}$ to be the full subcategory of **Cat** consisting of those objects I such that for all cofibrant-fibrant objects $\mathbf{X}$ of $\mathbf{M}^I$, the canonical map

$$hocolim_I F\mathbf{X} \to Fhocolim_I \mathbf{X}$$

is a weak homotopy equivalence. Then one has $<\mathbf{M},F>=<\mathbf{M},F>_{cf}$ if F preserves weak equivalences between cofibrant objects.

We remind the reader ([8], 19.6.1) that a functor $F : I \to J$ between small categories is **homotopy final** if for every $j \in ObJ$, $N(j \downarrow F)$ is weakly contractible. The composite of two homotopy final functors is homotopy final.

Proposition 1.2. *Let $I \to J$ be a homotopy final functor. Suppose that F preserves weak equivalences between cofibrant objects. Then $I \in<\mathbf{M},F>$ implies $J \in<\mathbf{M},F>$. If every $\mathbf{X} : I \to \mathbf{M}$ taking cofibrant values is weakly equivalent to the restriction to I of some $\mathbf{Y} : J \to \mathbf{M}$ taking cofibrant values, then $J \in<\mathbf{M},F>$ implies $I \in<\mathbf{M},F>$.*

Proof. The first part follows from ([8], 19.6.7(1)). The second part is straightforward. □

Corollary 1.3. *Assume that F preserves weak equivalences between cofibrant objects. Then*
(a) *every small category with terminal object is in $<\mathbf{M},F>$, and*
(b) *$<\mathbf{M},F>$ is invariant under Morita equivalences of categories.*

Proof. For (a) one uses ([8], 19.6.8(1)). For (b) one uses the fact that if $f : I \to J$ is a Morita equivalence between small categories, then $f_! : \mathbf{M}^I \rightleftarrows \mathbf{M}^J : f^*$ is an adjoint equivalence. □

Lemma 1.4. *Suppose that $<\mathbf{M},F>$ contains all small categories with terminal object and that F preserves all small filtered colimits. Then every small filtered category is in $<\mathbf{M},F>$.*

Proof. Let I be a small filtered category and let $\mathbf{X} : I \to \mathbf{M}$ take cofibrant values. For each $i \in ObI$, let $\mathbf{X}/i : (I \downarrow i) \to \mathbf{M}$ be $\mathbf{X}/i(k, k \to i) = \mathbf{X}_k$. Then $i \mapsto hocolim_{(I\downarrow i)}\mathbf{X}/i$ is a functor $I \to \mathbf{M}$ and one has ([2], XII, 3.5)

$$hocolim_I \mathbf{X} \cong colim_I (hocolim_{(I\downarrow i)}\mathbf{X}/i)$$

The map $hocolim_I F\mathbf{X} \to F hocolim_I \mathbf{X}$ is then isomorphic to the composite map

$$colim_I(hocolim_{(I\downarrow i)}F\mathbf{X}/i) \to colim_I(Fhocolim_{(I\downarrow i)}\mathbf{X}/i) \cong F(colim_I(hocolim_{(I\downarrow i)}\mathbf{X}/i))$$

The maps $hocolim_{(I\downarrow i)}F\mathbf{X}/i \to Fhocolim_{(I\downarrow i)}\mathbf{X}/i$ are weak homotopy equivalences by assumption, and since weak homotopy equivalences are stable under filtered colimits, we are done. □

The following result is a variation on lemma 1.4.

Lemma 1.5. *Suppose that F preserves filtered colimits and let I be a small filtered category. Then for every $\chi : I \to < \mathbf{M}, F >$ one has $colim_I \chi \in < \mathbf{M}, F >$.*

Proof. Let $\mathcal{C} = colim_I \chi$ and let $\tau_i : \chi_i \to \mathcal{C}$ be the canonical map ($i \in ObI$). For every $c \in Ob\mathcal{C}$ one has

$$colim_I N(c \downarrow \tau_i)^{op} \cong N(c \downarrow \mathcal{C})^{op} \qquad (*)$$

Let $\mathbf{X} : \mathcal{C} \to \mathbf{M}$ take cofibrant values. Using ([8], 19.6.6(1)), iterated coends and formula $(*)$ we obtain

$$hocolim_{\mathcal{C}}\mathbf{X} \cong colim_I(hocolim_{\chi_i}\mathbf{X}/i)$$

where $\mathbf{X}/i$ is the composite $\chi_i \xrightarrow{\tau_i} \mathcal{C} \xrightarrow{\mathbf{X}} \mathbf{M}$. The rest is similar to the proof of lemma 1.4. □

Let $\alpha : F \Rightarrow G : \mathbf{M} \to \mathbf{S}$ be a simplicial natural transformation between simplicial functors. Let $\mathbf{M}[\alpha]$ be the full subcategory of $\mathbf{M}$ consisting of those cofibrant objects A for which α_A is a weak homotopy equivalence.

Lemma 1.6. *In the above notation, if $I \in < \mathbf{M}, F > \cap < \mathbf{M}, G >$ then for every $\mathbf{X} : I \to \mathbf{M}[\alpha]$, $hocolim_I \mathbf{X} \in \mathbf{M}[\alpha]$.*

Proposition 1.7. *Suppose that $\mathbf{M}$ is cofibrantly generated and that F preserves weak equivalences between cofibrant objects. Let $p : \mathbb{E} \to \mathbb{B}$ be a split opfibration between small categories. If $\mathbb{B} \in < \mathbf{M}, F >$ and all the fibre categories $\mathbb{E}_b$ ($b \in Ob\mathbb{B}$) belong to $< \mathbf{M}, F >$, then $\mathbb{E}$ belongs to $< \mathbf{M}, F >$.*

Proof. Let $\mathbf{X} : \mathbb{E} \to \mathbf{M}$ take cofibrant values. We have a commutative diagram

$$\begin{array}{ccc}
hocolim_{\mathbb{B}} hocolim_{\mathbb{E}_b} F\mathbf{X}_b & \longrightarrow & hocolim_{\mathbb{B}} F(hocolim_{\mathbb{E}_b}\mathbf{X}_b) \\
\downarrow & & \downarrow \\
 & & F(hocolim_{\mathbb{B}}(hocolim_{\mathbb{E}_b}\mathbf{X}_b)) \\
 & & \downarrow \\
hocolim_{\mathbb{E}} F\mathbf{X} & \longrightarrow & Fhocolim_{\mathbb{E}}\mathbf{X}
\end{array}$$

The top horizontal arrow is a weak homotopy equivalence since all the fibre categories $\mathbb{E}_b$ ($b \in Ob\mathbb{B}$) belong to $< \mathbf{M}, F >$. The left and bottom right vertical arrows are weak homotopy equivalences by theorem 3.1 and the hypothesis of F, and the top right vertical arrow is a weak homotopy equivalence since $\mathbb{B} \in < \mathbf{M}, F >$. □

Corollary 1.8. *Suppose that* $\mathbf{M}$ *is cofibrantly generated and that* F *preserves weak equivalences between cofibrant objects. Let* $J \in < \mathbf{M}, F >$, $\mathcal{T}$ *a small category with all finite products and* $\mathbf{X} : \mathcal{T} \to Set^J$ *a product-preserving functor. Then the category* $El\mathbf{X}$ *of elements of* $\mathbf{X}$ *is in* $< \mathbf{M}, F >$.

Proof. Recall that $El\mathbf{X} = \int_{\mathcal{T}\times J} D\mathbf{X}$ (see section 3). The composite map $El\mathbf{X} \to \mathcal{T} \times J \to J$ is a split opfibration. The fibre category $(El\mathbf{X})_j$ over $j \in ObJ$ has all finite products since $\mathbf{X}$ is product-preserving. In particular, it has a terminal object, so $(El\mathbf{X})_j \in < \mathbf{M}, F >$ by corollary 1.3(a). □

The next result is a sort of estimation on the size of $< \mathbf{M}, F >$.

Lemma 1.9. *Suppose that* $\mathbf{M}$ *is cofibrantly generated and that* F *preserves weak equivalences between cofibrant objects. Then* $< \mathbf{M}, F >= \mathbf{Cat}$ *if and only if* $< \mathbf{M}, F >$ *contains* Δ^{op} *and the class of discrete categories.*

Proof. To begin with, let I be a small category. Let $y : \Delta \to \mathbf{S}$ be the Yoneda functor. There is a functor $f : (y \downarrow NI)^{op} \to I$, $([n], i_0 \to ... \to i_n) \mapsto i_0$ which is homotopy final ([3], 30.4). One has $(y \downarrow NI)^{op} = \int_{\Delta^{op}} DNI$ (see section 3). To ease notation we let $\mathbb{E} = \int_{\Delta^{op}} DNI$.

Suppose now that $< \mathbf{M}, F >$ contains Δ^{op} and the class of discrete categories. Let I be a small category and $\mathbf{X} : I \to \mathbf{M}$ take cofibrant values. We have a commutative diagram

$$\begin{array}{ccc} hocolim_{\mathbb{E}} F\mathbf{X}f & \longrightarrow & hocolim_I F\mathbf{X} \\ \downarrow & & \downarrow \\ Fhocolim_{\mathbb{E}}\mathbf{X}f & \longrightarrow & Fhocolim_I \mathbf{X} \end{array}$$

in which the horizontal arrows are weak homotopy equivalences. For each $[n] \in Ob\Delta$, the fibre category $\mathbb{E}_n$ over $[n]$ is discrete, hence by proposition 1.7 $\mathbb{E} \in < \mathbf{M}, F >$, which implies that the left vertical arrow in the preceding diagram is a weak homotopy equivalence. □

Definition 1.10. *Let* A *be an object of* $\mathbf{M}$. *We put* $< \mathbf{M}, A >=< \mathbf{M}, Map(A, -) >$.

Lemma 1.11. *Let* A *and* A' *be two objects of* $\mathbf{M}$. *If there is a simplicial homotopy equivalence between them, then* $< \mathbf{M}, A >=< \mathbf{M}, A' >$.

Proof. Let $A \to A'$ be a simplicial homotopy equivalence and let $\mathbf{X} : I \to \mathbf{M}$ take cofibrant values. Form the commutative diagram

$$\begin{array}{ccc} hocolim_I Map(A', \mathbf{X}) & \longrightarrow & Map(A', hocolim_I \mathbf{X}) \\ \downarrow & & \downarrow \\ hocolim_I Map(A, \mathbf{X}) & \longrightarrow & Map(A, hocolim_I \mathbf{X}) \end{array}$$

The vertical arrows are weak homotopy equivalences by ([8], 9.6.8) and the lemma follows. □

To illustrate the lemma, if I is a small category having either initial or terminal object, then $< \mathbf{S}, (\_)^{NI} >= \mathbf{Cat}$ and $< \mathbf{M}, A >=< \mathbf{M}, NI \otimes A >$.

Example 1.12. Let I be a small category, $i \in ObI$ and $[n] \in Ob\Delta$. Let $R_{i,n} \in \mathbf{S}^I$ be $j \mapsto \coprod_{i \to j} \Delta[n]$. Since $Map(R_{i,n}, \phi) \cong \phi(i)^{\Delta[n]}$ for all $\phi \in \mathbf{S}^I$, one has $< \mathbf{S}^I, R_{i,n} >= \mathbf{Cat}$.

2. Homotopy sifted categories and homotopy sifted-flat functors

For a set A, we denote by DA the discrete category on A. We let $Finset$ denote the category of finite sets and $Finset^+$ the category of finite nonempty sets.

Definition 2.1. *A small category is said to be* **homotopy sifted** *if it belongs to* $\bigcap_{A\in Finset} < \mathbf{S}^{DA}, lim >$.

Observe that $I \in < \mathbf{S}^{D\emptyset}, lim >$ if and only if I is weakly contractible if and only if $I \to 1$ is homotopy final, see ([8], 18.1.6). Hence a homotopy sifted category is weakly contractible, in particular it is nonempty. Observe also that

$$\bigcap_{A\in Finset^+} < \mathbf{S}^{DA}, lim > = < \mathbf{S}^{D\{1,2\}}, lim >$$

The category Δ^{op} is homotopy sifted: it is clearly weakly contractible, and then use ([8], 18.7.5) plus the fact that the geometric realization is isomorphic to the diagonal.

Proposition 2.2. *A category I is homotopy sifted if and only if for every finite set A, the diagonal functor $d : I \to I^{DA}$ is homotopy final.*

The previous observations imply that it suffices to prove the statement: $I \in < \mathbf{S}^{D\{1,2\}}, lim >$ if and only if the diagonal $d : I \to I \times I$ is homotopy final. This is ([11], Theorem 4.4).

Proof. "⇒" We have to show that for every pair i, j of objects of I, $N((i, j) \downarrow d)$ is weakly contractible. We will show that $N((i, j) \downarrow d)^{op}$ is weakly contractible. Let $cst : Set \to \mathbf{S}$ be the constant simplicial set functor and let $y_I : I^{op} \to Set^I$ be the Yoneda functor. By ([8], 19.6.11 and 19.6.6(1)) we have

$$N((i,j) \downarrow d)^{op} \cong cst^{I\times I} y_{I\times I}((i,j)) \otimes_{I\times I} N(? \downarrow d)^{op} \cong hocolim_I cst^{I\times I} y_{I\times I}((i,j))d$$

But

$$hocolim_I cst^{I\times I} y_{I\times I}((i,j))d = hocolim_I (cst^I y_I(i) \times cst^I y_I(j))$$

and by hypothesis the map

$$hocolim_I (cst^I y_I(i) \times cst^I y_I(j)) \to hocolim_I cst^I y_I(i) \times hocolim_I cst^I y_I(j)$$

is a weak homotopy equivalence. Since $hocolim_I cst^I y_I(i) \cong N(i \downarrow I)^{op}$ ([8], 19.6.10) and $N(i \downarrow I)^{op}$ is weakly contractible, we are done.

"⇐" Let $-\Box- : \mathbf{S}^I \times \mathbf{S}^I \to \mathbf{S}^{I\times I}$ be $\mathbf{X}\Box\mathbf{Y}_{(i,j)} = \mathbf{X}_i \times \mathbf{Y}_j$. Then the composite map

$$hocolim_I (\mathbf{X}\Box\mathbf{Y})d \to hocolim_{I\times I} \mathbf{X}\Box\mathbf{Y} \cong hocolim_I \mathbf{X} \times hocolim_I \mathbf{Y}$$

is a weak homotopy equivalence by assumption, and one has $\mathbf{X}\times\mathbf{Y} = (\mathbf{X}\Box\mathbf{Y})d$. □

Observe that for every $A \in Finset$, $< \mathbf{S}^{DA}, lim > = < \mathbf{S}^{DA}, holim >$, where $holim$ stands for the homotopy limit functor ([8], Chapter 18). Every small category with finite coproducts is homotopy sifted, cf. [11]: for, the constant functor $cst : I \to I^{D\{1,2\}}$ is now a right adjoint, hence homotopy final, but cst is isomorphic to the diagonal $I \to I \times I$. Every small category with terminal object is homotopy sifted by corollary 1.3(a).

As observed in [11], if Δ_1 is the 1-truncation of Δ, then Δ_1^{op} is not homotopy sifted, although it is weakly contractible. Indeed, write Δ_1^{op} as $\{B \overset{i}{\to} A \overset{f,g}{\rightrightarrows} B\}$ where $fi = gi = 1_B$. If Δ_1^{op} were homotopy sifted then by proposition 2.2 $((A, A) \downarrow d)$ would be in particular connected; here d is the diagonal functor. However, one can see that the objects $(A, A = A = A)$ and $(A, A \overset{f}{\to} B \overset{g}{\leftarrow} A)$ of $((A, A) \downarrow d)$ cannot be connected by a zig-zag of arrows.

Corollary 2.3. *A category is homotopy sifted if and only if it is homotopy final in its free completion under finite coproducts.*

Proof. For a small category I, we denote by $FamI$ its free completion under finite coproducts and by $\iota : I \to FamI$ the inclusion functor.

"$\Rightarrow$" Let I be homotopy sifted and let $(A, (i_a)_{a\in A}) \in ObFamI$. One has

$$((A, (i_a)_{a\in A}) \downarrow \iota) \cong ((i_a)_{a\in A} \downarrow d)$$

where $d : I \to I^{DA}$ is the diagonal.

"$\Leftarrow$" First note that $(\emptyset \downarrow \iota) \cong I$, where $\emptyset$ is the initial object of $FamI$. Then use the fact that $FamI$ is homotopy sifted and proposition 1.2 applied to ι. □

Lemmas 1.4 and 1.5 yield

Corollary 2.4. *Every small filtered category is homotopy sifted. The class of homotopy sifted categories is closed under filtered colimits.*

The first part of the previous corollary was noticed in [11]. Corollary 1.8 yields

Corollary 2.5. *Let $\mathcal{T}$ be a small category with all finite products and $\mathbf{X} : \mathcal{T} \to \mathbf{S}$ a product-preserving functor. Then the category of elements of $\mathbf{X}$ is homotopy sifted.*

Proposition 2.6. *A category I is homotopy sifted if and only if the composite functor $\mathbf{S}^I \overset{El}{\to} \mathbf{Cat} \overset{(\_)^{op}}{\to} \mathbf{Cat} \overset{N}{\to} \mathbf{S}$ preserves finite products up to homotopy. By this we mean that $N(Elcst1)^{op} \to 1$ is a weak homotopy equivalence and for every pair of objects $\mathbf{X}$,$\mathbf{Y}$ of $\mathbf{S}^I$, the canonical map*

$$N(El(\mathbf{X} \times \mathbf{Y}))^{op} \to N(El\mathbf{X})^{op} \times N(El\mathbf{Y})^{op}$$

is a weak homotopy equivalence.

Proof. One has $N(Elcst1)^{op} \cong N\Delta \times NI^{op}$, therefore I is weakly contractible if and only if $N(Elcst1)^{op} \to 1$ is a weak homotopy equivalence. Let now $\mathbf{X}, \mathbf{Y} \in \mathbf{S}^I$. By corollary 3.8 we have a weak homotopy equivalence

$$hocolim_{I\times\Delta^{op}} N^{op} D\mathbf{X} \to N(El\mathbf{X})^{op}$$

But $hocolim_{I\times\Delta^{op}} N^{op} D\mathbf{X} \cong N\Delta \times hocolim_I \mathbf{X}$ since, in general, if X is a simplicial set then $hocolim_{\Delta^{op}} NDX \cong N\Delta \times X$. We have then a commutative diagram

$$\begin{array}{ccc} N\Delta \times hocolim_I(\mathbf{X} \times \mathbf{Y}) & \longrightarrow & N(El(\mathbf{X} \times \mathbf{Y}))^{op} \\ \downarrow & & \downarrow \\ N\Delta \times hocolim_I \mathbf{X} \times N\Delta \times hocolim_I \mathbf{Y} & \longrightarrow & N(El\mathbf{X})^{op} \times N(El\mathbf{Y})^{op} \end{array}$$

in which the horizontal arrows are weak homotopy equivalences. The proposition follows. □

We also record the straightforward

Lemma 2.7. *Let $\mathbf{M}$ be a simplicial model category and $F, G : \mathbf{M} \to \mathbf{S}$ be simplicial functors. Let $F \times G : \mathbf{M} \to \mathbf{S}$ be $(F \times G)A = FA \times GA$. If $I \in < \mathbf{S}^{D\{1,2\}}, lim >$ then $I \in < \mathbf{M}, F > \cap < \mathbf{M}, G >$ implies $I \in < \mathbf{M}, F \times G >$.*

Let now I be a small category and $\phi \in \mathbf{S}^I$ a cofibrant object. Then we have an adjoint pair

$$\phi \otimes_{I^{op}} - : \mathbf{S}^{I^{op}} \rightleftarrows \mathbf{S} : (\_)^{\phi}$$

where $(X^{\phi})_i = X^{\phi(i)}$. The functor $\phi \otimes_{I^{op}} -$ preserves weak equivalences ([8], 18.4.4(1)). The next definition was inspired by the notion of sifted-flat functor of [1].

Definition 2.8. *ϕ is* **homotopy sifted-flat** *if the functor $\phi \otimes_{I^{op}} -$ preserves finite products up to homotopy. By this mean that*

(a) the map $(colim_I \phi \cong)\phi \otimes_{I^{op}} cst1 \to 1$ is a weak homotopy equivalence, and

(b) for every pair of objects $\mathbf{X}$,$\mathbf{Y}$ of $\mathbf{S}^{I^{op}}$, the canonical map

$$\phi \otimes_{I^{op}} (\mathbf{X} \times \mathbf{Y}) \to (\phi \otimes_{I^{op}} \mathbf{X}) \times (\phi \otimes_{I^{op}} \mathbf{Y})$$

is a weak homotopy equivalence.

Thus, given a small category I, I^{op} is homotopy sifted if and only if $N(I \downarrow ?)$ is homotopy sifted-flat.

For the proof of some of the next results we recall the following formulas.

(1) For every small category J and every $\mathbf{Z} : J \to \mathbf{S}^{I^{op}}$ we have

$$hocolim_J \phi \otimes_{I^{op}} \mathbf{Z} \cong \phi \otimes_{I^{op}} hocolim_J \mathbf{Z}$$

(2) Let J_1 and J_2 be small categories. Let $-\square- : (\mathbf{S}^{I^{op}})^{J_1} \times (\mathbf{S}^{I^{op}})^{J_2} \to (\mathbf{S}^{I^{op}})^{J_1 \times J_2}$ be $\mathbf{Z}\square\mathbf{T}((j_1, j_2)) = \mathbf{Z}(j_1) \times \mathbf{T}(j_2)$. Then

$$hocolim_{J_1 \times J_2} \mathbf{Z}\square\mathbf{T} \cong hocolim_{J_1} \mathbf{Z} \times hocolim_{J_2} \mathbf{T}$$

(3) If one fixes $\mathbf{X} \in \mathbf{S}^{I^{op}}$ and considers the functor $- \otimes_{I^{op}} \mathbf{X} : \mathbf{S}^I \to \mathbf{S}$, then the obvious analogue of (1) holds.

Lemma 2.9. (a) *Let I be a small category. The class of homotopy sifted-flat functors belonging to $\mathbf{S}^I$ is closed under homotopy sifted homotopy colimits.*

(b) *If $I \to J$ is a Morita equivalence between small categories, then the isomorphism class of homotopy sifted-flat functors belonging to $\mathbf{S}^I$ is in bijection with the isomorphism class of homotopy sifted-flat functors belonging to $\mathbf{S}^J$.*

Proof. (a) Let J be homotopy sifted and $\mathbf{Z} : J \to \mathbf{S}^I$ take homotopy sifted-flat values. One has, by (3) and ([8], 18.1.6),

$$hocolim_J \mathbf{Z} \otimes_{I^{op}} cst1 \cong hocolim_J (\mathbf{Z} \otimes_{I^{op}} cst1) \cong hocolim_J colim_I \mathbf{Z} \to hocolim_J cst1 \cong NJ^{op}$$

in which the only non-isomorphism is a weak equivalence by hypothesis. Therefore $hocolim_J \mathbf{Z}$ satisfies definition 2.8(a). The rest is a direct consequence of formulas (2),(3) above with the help of proposition 2.2. Another proof can be given using lemma 1.6. Part (b) is clear. □

Proposition 2.10. *The followings are equivalent for a cofibrant object ϕ of $\mathbf{S}^I$:*

(1) ϕ is homotopy sifted-flat,

(2) the map $\phi \otimes_{I^{op}} cst1 \to 1$ is a weak homotopy equivalence, and for every pair of representable functors $R_1, R_2 \in \mathbf{S}^{I^{op}}$, the canonical map

$$\phi \otimes_{I^{op}} (R_1 \times R_2) \to (\phi \otimes_{I^{op}} R_1) \times (\phi \otimes_{I^{op}} R_2)$$

is a weak homotopy equivalence.

Proof. (2)⇒(1). Let $\mathbf{X}, \mathbf{Y} \in \mathbf{S}^{I^{op}}$. By ([5], Proposition 2.9) the map $hocolim_{El\mathbf{X}}[I^{op}, \mathbf{X}] \to \mathbf{X}$ is a weak equivalence, so that the map

$$hocolim_{El\mathbf{X}}[I^{op}, \mathbf{X}] \times hocolim_{El\mathbf{Y}}[I^{op}, \mathbf{Y}] \to \mathbf{X} \times \mathbf{Y}$$

is a weak equivalence. To ease notation let $A = hocolim_{El\mathbf{X}}[I^{op}, \mathbf{X}]$ and $B = hocolim_{El\mathbf{Y}}[I^{op}, \mathbf{Y}]$. We have a commutative diagram

$$\begin{array}{ccc} \phi \otimes_{I^{op}} (A \times B) & \longrightarrow & (\phi \otimes_{I^{op}} A) \times (\phi \otimes_{I^{op}} B) \\ \downarrow & & \downarrow \\ \phi \otimes_{I^{op}} (\mathbf{X} \times \mathbf{Y}) & \longrightarrow & (\phi \otimes_{I^{op}} \mathbf{X}) \times (\phi \otimes_{I^{op}} \mathbf{Y}) \end{array}$$

in which the vertical arrows are weak homotopy equivalences. The top horizontal arrow is isomorphic to the composite

$\phi \otimes_{I^{op}} hocolim_{El\mathbf{X}\times El\mathbf{Y}}[I^{op},\mathbf{X}]\Box[I^{op},\mathbf{Y}] \cong hocolim_{El\mathbf{X}\times El\mathbf{Y}}\phi\otimes_{I^{op}}([I^{op},\mathbf{X}]\Box[I^{op},\mathbf{Y}]) \to hocolim_{El\mathbf{X}\times El\mathbf{Y}}(\phi\otimes_{I^{op}}[I^{op},\mathbf{X}]\Box\phi\otimes_{I^{op}}[I^{op},\mathbf{Y}]) \cong hocolim_{El\mathbf{X}}\phi\otimes_{I^{op}}[I^{op},\mathbf{X}] \times hocolim_{El\mathbf{Y}}\phi\otimes_{I^{op}}[I^{op},\mathbf{Y}]$. To finish the proof it suffices to show that the map

$$\phi\otimes_{I^{op}}([I^{op},\mathbf{X}]\Box[I^{op},\mathbf{Y}]) \to \phi\otimes_{I^{op}}[I^{op},\mathbf{X}]\Box\phi\otimes_{I^{op}}[I^{op},\mathbf{Y}]$$

is a weak equivalence in $\mathbf{S}^{El\mathbf{X}\times El\mathbf{Y}}$. Evaluating this map at an object $(x,y)\in El\mathbf{X}\times El\mathbf{Y}$ gives the canonical map

$$\phi\otimes_{I^{op}}([I^{op},\mathbf{X}](x)\times[I^{op},\mathbf{Y}](y)) \to (\phi\otimes_{I^{op}}[I^{op},\mathbf{X}](x))\times(\phi\otimes_{I^{op}}[I^{op},\mathbf{Y}](y))$$

which is a weak homotopy equivalence since $[I^{op},\mathbf{X}](x)$ and $[I^{op},\mathbf{Y}](y)$ are representable functors. □

Lemma 2.11. *Let $\phi\in\mathbf{S}^I$ be cofibrant. If the category of elements of ϕ is homotopy sifted then ϕ is homotopy sifted-flat.*

Proof. By ([5], Proposition 2.9) the map $hocolim_{El\phi}[I,\phi]\to\phi$ is a weak equivalence, therefore for every $\mathbf{X}\in\mathbf{S}^{I^{op}}$ the map

$$hocolim_{El\phi}([I,\phi]\otimes_{I^{op}}\mathbf{X}) \cong (hocolim_{El\phi}[I,\phi])\otimes_{I^{op}}\mathbf{X} \to \phi\otimes_{I^{op}}\mathbf{X}$$

is a weak homotopy equivalence. In particular, the map $hocolim_{El\phi}[I,\phi]\otimes_{I^{op}}cst1 \to \phi\otimes_{I^{op}}cst1$ is a weak homotopy equivalence. The functor $[I,\phi]\otimes_{I^{op}}cst1 : El\phi\to\mathbf{S}$ is given by $([n],i,x\in\phi(i)_n)\mapsto R_{i,n}\otimes_{I^{op}}cst1\cong\Delta[n]$, where $R_{i,n}$ was defined in example 1.12. There is a weak equivalence $[I,\phi]\otimes_{I^{op}}cst1\to cst1$, hence we have a weak homotopy equivalence $hocolim_{El\phi}[I,\phi]\otimes_{I^{op}}cst1 \to hocolim_{El\phi}cst1 \cong N(El\phi)^{op}$. It follows that definition 2.8(a) is satisfied.

Let now $\mathbf{X},\mathbf{Y}\in\mathbf{S}^{I^{op}}$. We have a commutative diagram

$$\begin{array}{ccc}
hocolim_{El\phi}([I,\phi]\otimes_{I^{op}}(\mathbf{X}\times\mathbf{Y})) & \longrightarrow & hocolim_{El\phi}([I,\phi]\otimes_{I^{op}}\mathbf{X}\times[I,\phi]\otimes_{I^{op}}\mathbf{Y}) \\
\downarrow & & \downarrow \\
 & & (hocolim_{El\phi}[I,\phi])\otimes_{I^{op}}\mathbf{X}\times(hocolim_{El\phi}[I,\phi])\otimes_{I^{op}}\mathbf{Y} \\
 & & \downarrow \\
\phi\otimes_{I^{op}}(\mathbf{X}\times\mathbf{Y}) & \longrightarrow & (\phi\otimes_{I^{op}}\mathbf{X})\times(\phi\otimes_{I^{op}}\mathbf{Y})
\end{array}$$

The left and bottom right vertical arrows are weak homotopy equivalences, the top right vertical arrow is a weak homotopy equivalence by hypothesis.

Consider the map

$$[I,\phi]\otimes_{I^{op}}(\mathbf{X}\times\mathbf{Y}) \to [I,\phi]\otimes_{I^{op}}\mathbf{X}\times[I,\phi]\otimes_{I^{op}}\mathbf{Y}$$

is $\mathbf{S}^{El\phi}$. Evaluating it at an object $([n],i,x\in\phi(i)_n)$ of $El\phi$ gives a map isomorphic to $\Delta[n]\times\mathbf{X}_i\times\mathbf{Y}_i\to\Delta[n]\times\mathbf{X}_i\times\Delta[n]\times\mathbf{Y}_i$, which is a weak homotopy equivalence. Therefore the top horizontal map in the previous diagram is a weak homotopy equivalence. □

We don't know whether the converse holds in the preceding lemma.

Proposition 2.12. *Let I be a small category with finite products. A cofibrant object ϕ of $\mathbf{S}^I$ is homotopy sifted-flat if and only if $\phi(1)$ is weakly contractible and for all objects i,j of I, the natural map $\phi(i\times j)\to\phi(i)\times\phi(j)$ is a weak homotopy equivalence.*

Proof. "⇒" Let $r : I \to \mathbf{S}^{I^{op}}$ be $r(i)(j) = \underset{j \to i}{\sqcup} \Delta[0]$. Then r preserves finite products and $\phi \otimes_{I^{op}} r(i) \cong \phi(i)$. The claim follows.

"⇐" For every object $\mathbf{X}$ of $\mathbf{S}^{I^{op}}$ we denote by $[I^{op}, \mathbf{X}]' : El\mathbf{X} \to \mathbf{S}^{I^{op}}$ the functor $(([n], i, x \in \mathbf{X}(i)_n) \mapsto (j \mapsto \underset{j \to i}{\sqcup} \Delta[0])$. There is a natural transformation $[I^{op}, \mathbf{X}] \Rightarrow [I^{op}, \mathbf{X}]'$, hence an induced weak equivalence $hocolim_{El\phi}[I^{op}, \phi] \to hocolim_{El\phi}[I^{op}, \phi]'$.

Let $\mathbf{X}, \mathbf{Y} \in \mathbf{S}^{I^{op}}$. To ease notation we put $A = hocolim_{El\mathbf{X}}[I^{op}, \mathbf{X}]$, $B = hocolim_{El\mathbf{Y}}[I^{op}, \mathbf{Y}]$, $A' = hocolim_{El\mathbf{X}}[I^{op}, \mathbf{X}]'$ and $B' = hocolim_{El\mathbf{Y}}[I^{op}, \mathbf{Y}]'$. We have a commutative diagram

$$\begin{array}{ccc} \phi \otimes_{I^{op}} (A' \times B') & \longrightarrow & (\phi \otimes_{I^{op}} A') \times (\phi \otimes_{I^{op}} B') \\ \uparrow & & \uparrow \\ \phi \otimes_{I^{op}} (A \times B) & \longrightarrow & (\phi \otimes_{I^{op}} A) \times (\phi \otimes_{I^{op}} B) \\ \downarrow & & \downarrow \\ \phi \otimes_{I^{op}} (\mathbf{X} \times \mathbf{Y}) & \longrightarrow & (\phi \otimes_{I^{op}} \mathbf{X}) \times (\phi \otimes_{I^{op}} \mathbf{Y}) \end{array}$$

in which the vertical arrows are weak homotopy equivalences. It suffices to prove that the top horizontal arrow is a weak homotopy equivalence. The proof is similar to (2)⇒(1) of proposition 2.10, so we shall only sketch it. The map

$$\phi \otimes_{I^{op}} ([I^{op}, \mathbf{X}]' \Box [I^{op}, \mathbf{Y}]') \to \phi \otimes_{I^{op}} [I^{op}, \mathbf{X}]' \Box \phi \otimes_{I^{op}} [I^{op}, \mathbf{Y}]'$$

is a weak equivalence in $\mathbf{S}^{El\mathbf{X} \times El\mathbf{Y}}$ since when evaluated at an object $(([m], i, x \in \mathbf{X}(i)_m), ([n], j, y \in \mathbf{Y}(j)_n)) \in El\mathbf{X} \times El\mathbf{Y}$ is (isomorphic to) the map $\phi(i \times j) \to \phi(i) \times \phi(j)$. □

3. Appendix: The Grothendieck construction and a result of Chacholski and Scherer

In this section we recall the Grothendieck construction and a result of W. Chacholski and J. Scherer [3].

Let I be a small category and let $\mathcal{F} : I \to \mathbf{Cat}$ be a functor. The **Grothendieck construction** on $\mathcal{F}$ is the category $\int_I \mathcal{F}$ defined as follows. An object of $\int_I \mathcal{F}$ is a pair (i, a) consisting of $i \in ObI$ and $a \in Ob\mathcal{F}_i$. An arrow from (i, a) to (j, b) is a pair (f, u) consisting of an arrow $f : i \to j$ in I and an arrow $u : \mathcal{F}_f(a) \to b$ in $\mathcal{F}_j$. Composition is defined by the formula

$$(g, v)(f, u) = (gf, v\mathcal{F}_g(u))$$

where $(f, u) : (i, a) \to (j, b)$ and $(g, v) : (j, b) \to (k, c)$ are two arrows of $\int_I \mathcal{F}$.

Let ${}^{op}\mathcal{F}$ be the composite functor $I \xrightarrow{\mathcal{F}} \mathbf{Cat} \xrightarrow{(\text{-})^{op}} \mathbf{Cat}$. Then $\int_I^{op} \mathcal{F}$ has the same objects as $\int_I \mathcal{F}$ and an arrow $(i, a) \to (j, b)$ is a pair (f, u), where $f : i \to j$ and $u : b \to \mathcal{F}_f(a)$.

There is a functor $p_{\mathcal{F}} : \int_I \mathcal{F} \to I$, $(i, a) \mapsto i$, and this is a Grothendieck opfibration. For each $i \in ObI$ there are functors $\mathcal{F}_i \to \int_I \mathcal{F}$ and this identifies the fibre of $p_{\mathcal{F}}$ above i with $\mathcal{F}_i$.

As an example, let $F : I \to Set$ be a functor. The category ElF of elements of F, as defined at the beginning of this article, is precisely the category $\int_I DF$, where $D : Set \to \mathbf{Cat}$ is the discrete category functor. One has $\int_I Dcst1 \cong I$. If $y : I^{op} \to Set^I$ is the Yoneda functor, then $(y \downarrow F) = (\int_I DF)^{op}$.

Let now $p : \mathbb{E} \to \mathbb{B}$ a split opfibration between small categories. We denote by $\mathbb{E}_b$ the fibre category over $b \in Ob\mathbb{B}$ and by $\iota_b : \mathbb{E}_b \to \mathbb{E}$ the natural functor. Let $\mathbf{M}$ be an arbitrary simplicial model category and $\mathbf{X} : \mathbb{E} \to \mathbf{M}$ a functor. We obtain a functor

$$\mathbb{B} \to (\mathbf{Cat} \downarrow \mathbf{M}), \qquad b \mapsto (\mathbb{E}_b, \mathbf{X}_b := \mathbf{X}\iota_b)$$

Therefore there is a natural map

$$hocolim_{\mathbb{B}} hocolim_{\mathbb{E}_b} \mathbf{X}_b \to hocolim_{\mathbb{E}} \mathbf{X}$$

The next theorem is ([3], Theorem 26.8) translated into the language of model categories.

Theorem 3.1. *Let $\mathbf{M}$ be a cofibrantly generated simplicial model category, $p : \mathbb{E} \to \mathbb{B}$ a split opfibration between small categories and $\mathbf{X} : \mathbb{E} \to \mathbf{M}$ a functor taking cofibrant values. Then the natural map*

$$hocolim_{\mathbb{B}} hocolim_{\mathbb{E}_b} \mathbf{X}_b \to hocolim_{\mathbb{E}} \mathbf{X}$$

is a weak equivalence.

We shall prove this theorem in a succession of lemmas.

Lemma 3.2. *It suffices to prove theorem 3.1 in the case when $\mathbf{X}$ is cofibrant in $\mathbf{M}^{\mathbb{E}}$.*

Proof. Let $\widetilde{\mathbf{X}} \to \mathbf{X}$ be a cofibrant approximation to $\mathbf{X}$ in $\mathbf{M}^{\mathbb{E}}$. Then for every $b \in Ob\mathbb{B}$ we have a map $\widetilde{\mathbf{X}}_b \to \mathbf{X}_b$ which is natural in b. Therefore we have a commutative diagram

$$\begin{array}{ccc} hocolim_{\mathbb{B}} hocolim_{\mathbb{E}_b} \widetilde{\mathbf{X}}_b & \longrightarrow & hocolim_{\mathbb{E}} \widetilde{\mathbf{X}} \\ \downarrow & & \downarrow \\ hocolim_{\mathbb{B}} hocolim_{\mathbb{E}_b} \mathbf{X}_b & \longrightarrow & hocolim_{\mathbb{E}} \mathbf{X} \end{array}$$

in which the top horizontal map is a weak equivalence by assumption and the vertical maps are weak equivalences by ([8], 18.5.3(1)). □

Lemma 3.3. ([9], Corollary 1.4.4(b)) *Let $F_i : \mathbf{M} \rightleftarrows \mathbf{N} : G_i$, $i \in \{1, 2\}$, be a Quillen pair between model categories. To give a natural transformation $F_1 \Rightarrow F_2$ which is a weak equivalence on cofibrant objects is to give a natural transformation $G_2 \Rightarrow G_1$ which is a weak equivalence on fibrant objects.*

The next result is standard.

Lemma 3.4. (a) *Let $\mathbf{M}$ be a cofibrantly generated simplicial model category and I an arbitrary small category. Then for every cofibrant object $\mathbf{X} \in \mathbf{M}^I$, the natural map*

$$hocolim_I \mathbf{X} \to colim_I \mathbf{X}$$

is a weak equivalence.

(b) *Let $\mathbf{M}$ be a cofibrantly generated simplicial model category, $F : \mathcal{C} \to \mathcal{D}$ an arbitrary functor between small categories and $\mathbf{X} \in \mathbf{M}^{\mathcal{C}}$ a cofibrant object. Then the natural map*

$$hocolim_{\mathcal{C}} \mathbf{X} \to hocolim_{\mathcal{D}} F_! \mathbf{X}$$

is a weak equivalence.

Proof. (a) The functor $hocolim_I$ has a right adjoint R given by $(RA)_i = A^{N(i\downarrow I)^{op}}$. $(hocolim_I, R)$ is a Quillen pair by ([8], 18.5.1(1) and 11.6.3). There is a natural transformation $cst \Rightarrow R$, where cst is the constant functor, which is a weak equivalence on fibrant objects by ([8], 9.5.18(2) and 9.5.16). Then apply lemma 3.3.

(b) We have a diagram of Quillen pairs

$$\begin{array}{ccc} \mathbf{M}^{\mathcal{C}} & \underset{F^*}{\overset{F_!}{\rightleftarrows}} & \mathbf{M}^{\mathcal{D}} \\ R_1 \uparrow\downarrow hocolim_{\mathcal{C}} & & R_2 \uparrow\downarrow hocolim_{\mathcal{D}} \\ \mathbf{M} & = & \mathbf{M} \end{array}$$

There is a natural transformation $F^*R_2 \Rightarrow R_1$ constructed as follows: one has $(F^*R_2A)_c = A^{N(Fc\downarrow\mathcal{D})^{op}}$ and there is a natural map $N(?\downarrow\mathcal{C})^{op} \to N(p?\downarrow\mathcal{D})^{op}$. Therefore, by lemma 3.3 one has an induced natural transformation $hocolim_{\mathcal{C}} \Rightarrow hocolim_{\mathcal{D}}F_!$. Now apply part (a). □

Lemma 3.5. ([4], Proposition 3.1.13) *Let* $\mathbf{M}$ *be a cofibrantly generated model category. Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between small categories and let* $d \in Ob\mathcal{D}$*. Let* $q : (F \downarrow d) \to \mathcal{C}$ *be the projection. Then the functor* $q^* : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}^{(F\downarrow d)}$, $\mathbf{X} \mapsto \mathbf{X}q$, *is a left Quillen functor.*

The next result is a weaker form of ([6], 9.8).

Lemma 3.6. *Let* $\mathbf{M}$ *be a cofibrantly generated simplicial model category. Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between small categories and let* $d \in Ob\mathcal{D}$*. Let* $q : (F \downarrow d) \to \mathcal{C}$ *be the projection. Then for every cofibrant object* $\mathbf{X} \in \mathbf{M}^{\mathcal{C}}$*, the natural map*

$$hocolim_{\mathcal{D}} hocolim_{(F\downarrow d)} \mathbf{X}q \to hocolim_{\mathcal{C}} \mathbf{X}$$

is a weak equivalence.

Proof. Let $(F_!^h\mathbf{X})_d = hocolim_{(F\downarrow d)}\mathbf{X}q$, so that $(F_!^h\mathbf{X}) \in \mathbf{M}^{\mathcal{D}}$. By lemmas 3.5 and 3.4$(a)$ the natural map $(F_!^h\mathbf{X})_d \to (F_!\mathbf{X})_d = colim_{(F\downarrow d)}\mathbf{X}q$ is a weak equivalence (between cofibrant objects), therefore the map $hocolim_{\mathcal{D}}(F_!^h\mathbf{X}) \to hocolim_{\mathcal{D}}(F_!\mathbf{X})$ is a weak equivalence. This, combined with lemma 3.4(b), proves the result. □

Lemma 3.7. *Let* $\mathbf{M}$ *be a cofibrantly generated simplicial model category,* $p : \mathbb{E} \to \mathbb{B}$ *a split opfibration between small categories and* $\mathbf{X} \in \mathbf{M}^{\mathbb{E}}$ *a cofibrant object. Then the natural map*

$$hocolim_{\mathbb{B}} hocolim_{\mathbb{E}_b} \mathbf{X}_b \to hocolim_{\mathbb{E}} \mathbf{X}$$

is a weak equivalence.

Proof. For every $b \in Ob\mathbb{B}$ there is an adjunction $L_b : (p \downarrow b) \rightleftarrows \mathbb{E}_b : R_b$. Let $q : (p \downarrow b) \to \mathbb{E}$ be the projection. For every $b \in Ob\mathbb{B}$ we have $\mathbf{X}qR_b = \mathbf{X}_b$, hence the map

$$hocolim_{\mathbb{E}_b} \mathbf{X}_b \to hocolim_{(p\downarrow b)} \mathbf{X}q$$

is a weak equivalence (between cofibrant objects). Therefore the induced map on homotopy colimits is a weak equivalence, so that by lemma 3.6 the required map is a weak equivalence. □

Theorem 3.1 follows now from lemmas 3.2 and 3.7. Using ([8], 18.1.6) one obtains

Corollary 3.8. ([12], Theorem 1.2) *Let I be a small category and let $\mathcal{F} : I \to$ **Cat** be a functor. Then there is a weak homotopy equivalence*

$$hocolim_I N^{op}\mathcal{F} \to N(\int_I \mathcal{F})^{op}$$

The sign "op" appears because Hirschhorn's definition of the nerve functor [8] is not the same as Thomason's.

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