GEOMETRY OF THE SPACE OF MULTISYMPLECTIC FORMS

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Abstract. We study the geometry of the space of real 3-forms in dimensions 6 and 7. To be more precise, we calculate the closure of orbits of these forms under the action of the general linear group. Moreover, the question of the number of components and convexity of each type is settled.

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1. Introduction

In this paper we address the question of the geometric structure of 3-forms in dimensions 6 and 7. We calculate the closure of each orbit of these forms under the action of the general linear group.

The analogous task is trivial when it comes to 2-forms. On $\mathbb{R}^n$ we have $[n/2]+1$ different types of 2-forms. Let $\alpha_1, \dots, \alpha_n$ be a basis if $\mathbb{R}^{n*}$. We denote $M_i \subset \Lambda^2\mathbb{R}^{n*}$, $i = 0, 2, \dots, 2[n/2]$ the orbit under the action of $GL(n;\mathbb{R})$ containing the 2-form

$$\omega_i = \alpha_1 \wedge \alpha_2 + \cdots + \alpha_{2i-1} \wedge \alpha_{2i}.$$

If $n = 2k$, then ω_k is a symplectic form. In this paper we shall use the notation $\mathcal{A} \to \mathcal{B}$ whenever $\mathcal{B} \subset \overline{\mathcal{A}}$, i. e. $\mathcal{B}$ is contained in the closure of $\mathcal{A}$. Then we have the diagram

$$\mathcal{M}_{2[n/2]} \longrightarrow \mathcal{M}_{2[n/2]-2} \longrightarrow \cdots \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_0 = \{0\}.$$

However, in the case of 3-forms in dimensions 6 and 7 the problem becomes much harder and interesting.

Let us consider the 6-dimensional complex vector space $\mathbb{C}^6$. The group $GL(6;\mathbb{C})$ operates in a natural way on $\mathbb{C}^6$, and there is an induced operation on $\Lambda^3\mathbb{C}^{6*}$. Under this operation $\Lambda^3\mathbb{C}^{6*}$ decomposes into 5 orbits. We shall denote them by the roman numerals I, ..., V. They can be described by their representatives. Let us take a basis $\beta_1, \dots, \beta_6$ of $\mathbb{C}^{6*}$. Then the representatives have the form

$$\text{(I)} \quad s_1^{\mathbb{C}} = 0,$$

$$\text{(II)} \quad s_2^{\mathbb{C}} = \beta_1 \wedge \beta_2 \wedge \beta_3,$$

$$\text{(III)} \quad s_3^{\mathbb{C}} = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_1 \wedge \beta_4 \wedge \beta_5,$$

$$\text{(IV)} \quad r_0^{\mathbb{C}} = \beta_1 \wedge \beta_2 \wedge \beta_4 + \beta_1 \wedge \beta_3 \wedge \beta_5 + \beta_2 \wedge \beta_3 \wedge \beta_6,$$

$$\text{(V)} \quad r^{\mathbb{C}} = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5 \wedge \beta_6.$$

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Further information about these orbits we can find in the following table.

Type	$\mathbb{C}$-dimension of the isotropy group	$\mathbb{C}$-dimension
I	36	0
II	26	10
III	21	15
IV	17	19
V	16	20

Similar situation is with the 6-dimensional real vector space $\mathbb{R}^6$. Again the group $GL(6;\mathbb{R})$ operates in a natural way on $\mathbb{R}^6$, and there is an induced action of $GL(6;\mathbb{R})$ on $\Lambda^3\mathbb{R}^{6*}$. This time $\Lambda^3\mathbb{R}^{6*}$ decomposes into six orbits, which we denote $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{R}_0, \mathcal{R}_+, \mathcal{R}_-$. They can again be described by their representatives. Let $\alpha_1, \dots, \alpha_6$ be a basis of $\mathbb{R}^{6*}$. Then the above orbits have the following representatives.

$$
\begin{aligned}
(\mathcal{S}_1)\quad & s_1 = 0,\\
(\mathcal{S}_2)\quad & s_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3,\\
(\mathcal{S}_3)\quad & s_3 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5,\\
(\mathcal{R}_0)\quad & r_0 = \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + \alpha_1 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_6,\\
(\mathcal{R}_+)\quad & r_+ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6,\\
(\mathcal{R}_-)\quad & r_- = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.
\end{aligned}
$$

Further information about the orbit we can again find in the following table.

Type	$\mathbb{R}$-dimension of the isotropy group	$\mathbb{R}$-dimension
$\mathcal{S}_1$	36	0
$\mathcal{S}_2$	26	10
$\mathcal{S}_3$	21	15
$\mathcal{R}_0$	17	19
$\mathcal{R}_+$	16	20
$\mathcal{R}_-$	16	20

Using the results of [D2] we find that

$$\mathcal{S}_1 = \mathrm{I}, \quad \mathcal{S}_2 \subset \mathrm{II}, \quad \mathcal{S}_3 \subset \mathrm{III}, \quad \mathcal{R}_0 \subset \mathrm{IV}, \quad \mathcal{R}_+ \cup \mathcal{R}_- \subset \mathrm{V}\,.$$

Along the same lines we proceed in dimension 7. On the 7-dimensional complex vector space $\mathbb{C}^7$ we have the natural action of $GL(7;\mathbb{C})$. Then we have the induced action of $GL(7;\mathbb{C})$ on $\Lambda^3\mathbb{C}^{7*}$. According to this action $\Lambda^3\mathbb{C}^{7*}$ decomposes into 10 orbits. We shall denote them $\mathrm{I}, \dots, \mathrm{X}$. They can again be described by their representatives.

$$\omega_{\rm I} = 0, \tag{I}$$
$$\omega_{\rm II} = \beta_1 \wedge \beta_2 \wedge \beta_3, \tag{II}$$
$$\omega_{\rm III} = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_1 \wedge \beta_4 \wedge \beta_5, \tag{III}$$
$$\omega_{\rm IV} = \beta_1 \wedge \beta_2 \wedge \beta_4 + \beta_1 \wedge \beta_3 \wedge \beta_5 + \beta_2 \wedge \beta_3 \wedge \beta_6, \tag{IV}$$
$$\omega_{\rm V} = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5 \wedge \beta_6, \tag{V}$$
$$\omega_{\rm VI} = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_1 \wedge \beta_6 \wedge \beta_7, \tag{VI}$$
$$\omega_{\rm VII} = \beta_1 \wedge \beta_2 \wedge \beta_5 + \beta_1 \wedge \beta_3 \wedge \beta_6 + \beta_1 \wedge \beta_4 \wedge \beta_7 + \beta_2 \wedge \beta_3 \wedge \beta_4, \tag{VII}$$
$$\omega_{\rm VIII} = \beta_1 \wedge \beta_3 \wedge \beta_4 + \beta_2 \wedge \beta_5 \wedge \beta_6 + \beta_1 \wedge \beta_2 \wedge \beta_7, \tag{VIII}$$
$$\omega_{\rm IX} = \beta_1 \wedge \beta_2 \wedge \beta_5 + \beta_3 \wedge \beta_4 \wedge \beta_6 + \beta_1 \wedge \beta_3 \wedge \beta_7 + \beta_2 \wedge \beta_4 \wedge \beta_7, \tag{IX)}$$
$$\begin{aligned} \omega_{\rm X} = {}& \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5 \wedge \beta_6 + \beta_1 \wedge \beta_4 \wedge \beta_7 \\ &+ \beta_2 \wedge \beta_5 \wedge \beta_7 + \beta_3 \wedge \beta_6 \wedge \beta_7. \end{aligned} \tag{X}$$

Further information about these orbits we can find in the following table.

Type	$\mathbb{C}$-dimension of the isotropy group	$\mathbb{C}$-dimension
I	49	0
II	36	13
III	29	20
IV	24	25
V	23	26
VI	28	21
VII	21	28
VIII	18	31
IX	15	34
X	14	35

The group $GL(7;\mathbb{R})$ operates on $\Lambda^7\mathbb{R}^{7*}$, and this space decomposes into 14 orbits. To describe representatives, we choose a basis $\alpha_1, \dots, \alpha_7$ of $\mathbb{R}^{7*}$. First six orbits $\Sigma_1, \Sigma_2, \Sigma_3, \Omega_0, \Omega_+, \Omega_-$ have representatives determined by the same formulas as representatives of the orbits $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{R}_0, \mathcal{R}_+, \mathcal{R}_-$. (Only we must consider these forms on $\mathbb{R}^7$ and not on $\mathbb{R}^6$.) The next eight forms are defined by the following formulas.

$$(\Omega_1)\qquad \omega_1 = \alpha_1\wedge\alpha_2\wedge\alpha_7 + \alpha_1\wedge\alpha_3\wedge\alpha_4 + \alpha_2\wedge\alpha_5\wedge\alpha_6,$$
$$(\Omega_2)\qquad \omega_2 = \alpha_1\wedge\alpha_2\wedge\alpha_5 + \alpha_1\wedge\alpha_2\wedge\alpha_7 + \alpha_1\wedge\alpha_4\wedge\alpha_7 - \alpha_2\wedge\alpha_3\wedge\alpha_7 + \alpha_3\wedge\alpha_4\wedge\alpha_6 + \alpha_3\wedge\alpha_4\wedge\alpha_7,$$
$$(\Omega_3)\qquad \omega_3 = \alpha_1\wedge(\alpha_2\wedge\alpha_7 - \alpha_3\wedge\alpha_6 + \alpha_4\wedge\alpha_5),$$
$$(\Omega_4)\qquad \omega_4 = \alpha_1\wedge(\alpha_2\wedge\alpha_7 - \alpha_3\wedge\alpha_6 + \alpha_4\wedge\alpha_5) + \alpha_2\wedge\alpha_4\wedge\alpha_6,$$
$$(\Omega_5)\qquad \omega_5 = \alpha_1\wedge\alpha_2\wedge\alpha_3 - \alpha_1\wedge\alpha_4\wedge\alpha_5 + \alpha_1\wedge\alpha_6\wedge\alpha_7 + \alpha_2\wedge\alpha_4\wedge\alpha_6 + \alpha_2\wedge\alpha_5\wedge\alpha_7 + \alpha_3\wedge\alpha_4\wedge\alpha_7 - \alpha_3\wedge\alpha_5\wedge\alpha_6,$$
$$(\Omega_6)\qquad \omega_6 = \alpha_1\wedge\alpha_2\wedge\alpha_7 - \alpha_1\wedge\alpha_3\wedge\alpha_6 + \alpha_1\wedge\alpha_4\wedge\alpha_5 + \alpha_2\wedge\alpha_3\wedge\alpha_5 + \alpha_2\wedge\alpha_4\wedge\alpha_6,$$
$$(\Omega_7)\qquad \omega_7 = \alpha_1\wedge\alpha_2\wedge\alpha_5 + \alpha_1\wedge\alpha_3\wedge\alpha_6 + \alpha_1\wedge\alpha_4\wedge\alpha_7 + \alpha_2\wedge\alpha_3\wedge\alpha_7 - \alpha_2\wedge\alpha_4\wedge\alpha_6 + \alpha_3\wedge\alpha_4\wedge\alpha_5,$$
$$(\Omega_8)\qquad \omega_8 = \alpha_1\wedge\alpha_2\wedge\alpha_3 + \alpha_1\wedge\alpha_4\wedge\alpha_5 - \alpha_1\wedge\alpha_6\wedge\alpha_7 + \alpha_2\wedge\alpha_4\wedge\alpha_6 + \alpha_2\wedge\alpha_5\wedge\alpha_7 + +\alpha_3\wedge\alpha_4\wedge\alpha_7 - \alpha_3\wedge\alpha_5\wedge\alpha_6.$$

Here we get the table

Type	$\mathbb{R}$-dimension of the isotropy group	$\mathbb{R}$-dimension
Σ_1	49	0
Σ_2	36	13
Σ_3	29	20
Ω_0	24	25
Ω_+	23	26
Ω_-	23	26
Ω_1	18	31
Ω_2	15	34
Ω_3	28	21
Ω_4	21	28
Ω_5	14	35
Ω_6	18	31
Ω_7	15	34
Ω_8	14	35

Using again the results of [D2] we find

$$\Sigma_1 = \mathrm{I},\quad \Sigma_2\subset\mathrm{II},\quad \Sigma_3\subset\mathrm{III},\quad \Omega_0\subset\mathrm{IV},\quad \Omega_+\cup\Omega_-\subset\mathrm{V},$$
$$\Omega_3\subset\mathrm{VI},\quad \Omega_4\subset\mathrm{VII},\quad \Omega_1\cup\Omega_6\subset\mathrm{VIII},\quad \Omega_2\cup\Omega_7\subset\mathrm{IX},\quad \Omega_5\cup\Omega_8\subset\mathrm{X}\,.$$

We recall two invariants of a 3-form. If $\omega\in\Lambda^3V^*$, then there exists the smallest subspace $Z\subset V^*$ such that $\omega\in\Lambda^3Z$. The number $r=\dim Z$ is called the *rank* of the form ω. Furthermore, we define

$$\ker\omega = \{v\in V;\iota_v\omega = 0\},$$

where $(\iota_v\omega)(\cdot,\cdot) = \omega(v,\cdot,\cdot)$. Obviously $\dim\ker\omega = m - r$. If $\operatorname{rank}\omega = \dim V$, then the form ω is called *multisymplectic* or *regular*. We denote by $\Lambda^3_{ms}V^*$ the subset of Λ^3V^* consisting of multisymplectic forms. It is easy to see that this subset is open.

Let us consider unordered couples $\{p,q\}$ of nonnegative integers. We shall say that an unordered couple $\{r,s\}$ is subordinate to the unordered couple $\{p,q\}$ if either $r \leq p$ and $s \leq q$ or $s \leq p$ and $r \leq q$.

We shall now consider a symmetric bilinear form $B : V \times V \to \Lambda^m V^*$. Choosing a non-zero element $\mu \in \Lambda^m V^*$ we can write $B(v,v') = b_\mu(v,v')\mu$, where b_μ is a bilinear form with scalar values. Let us denote $\operatorname{Sign}_\mu B = \operatorname{Sign} b_\mu = (l,p,q)$, where $l = \dim(\ker b_\mu)$, and $\{p,q\}$ is the signature of b_μ. It is obvious that if $\tilde{\mu}$ is another non-zero element of $\Lambda^n V^*$, then either $\operatorname{Sign}_{\tilde{\mu}} B = (l,p,q)$ or $\operatorname{Sign}_{\tilde{\mu}} B = (l,q,p)$. This enables us to define $\operatorname{Sign} B$ by the formula

$$\operatorname{Sign} B = (l,\{p,q\}).$$

Let $\{B^{(n)}\}_{n=1}^\infty$ be a sequence such that $\operatorname{Sign} B^{(n)} = (l,\{p,q\})$ for every $n \in \mathbb{N}$. We shall assume that this sequence is convergent and that $\lim_{n\to\infty} B^{(n)} = B$. Let us choose a non-zero element $\mu \in \Lambda^m V^*$. Then we can write $B^{(n)} = b^{(n)}\mu$ and $B = b\mu$. There must be either $\operatorname{Sign} b^{(n)} = (l,p,q)$ for infinitely many $n \in \mathbb{N}$ or $\operatorname{Sign} b^{(n)} = (l,q,p)$ for infinitely many $n \in \mathbb{N}$. Let us assume that the first case occurs. This means that without a loss of generality we may assume that $\operatorname{Sign} b^{(n)} = (l,p,q)$ for all $n \in \mathbb{N}$. We denote $\operatorname{Sign} b = (l_0,p_0,q_0)$. Let us suppose now that $p < p_0$. Obviously, we can find a an p_0-dimensional subspace $W \subset V$ such that for every $w \in W$, $w \neq 0$ there is $b(w,w) > 0$. Technically, we can introduce on V an auxiliary scalar product and consider a unit sphere SW in W. Then, of course, $B(w,w) > 0$ for every $w \in SW$. Because $\lim_{n\to\infty} B^{(n)} = B$, we can easily see that for all n large enough we have $B^{(n)}(w,w) > 0$ for every $w \in SW$, which is a contradiction. This shows that $p \geq p_0$. Similarly we can prove that $q \geq q_0$. As a consequence we get then that $l \leq l_0$. We have thus proved the following lemma.

Lemma 1.1. *Let $\{B^{(n)}\}_{n=1}^\infty$ be a convergent sequence of bilinear forms $B^{(n)} : V \times V \to \Lambda^m V^*$ such that* $\operatorname{Sign} B^{(n)} = (l,\{p,q\})$ *for all $n \in \mathbb{N}$. Let us denote $B = \lim_{n\to\infty} B^{(n)}$ and* $\operatorname{Sign} B = (l_0,\{p_0,q_0\})$*. Then $l_0 \geq l$ and the couple $\{p_0,q_0\}$ is subordinate to the couple $\{p,q\}$.*

For our purposes we will frequently use the bilinear form

$$B_\omega(u,v) = (\iota_v\omega) \wedge (\iota_v\omega) \wedge \omega.$$

associated to any 3-form ω.

2. Dimension 6

The orbits in $\mathbb{C}^{6*}$ we denote I, …, V. According to [D2] we have

$$\overline{\mathrm{I}} = \mathrm{I}, \quad ,\overline{\mathrm{II}} = \mathrm{I}\cup\mathrm{II}, \quad \overline{\mathrm{III}} = \mathrm{I}\cup\mathrm{II}\cup\mathrm{III},$$
$$\overline{\mathrm{IV}} = \mathrm{I}\cup\mathrm{II}\cup\mathrm{III}, \quad \overline{\mathrm{V}} = \mathrm{I}\cup\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}\cup\mathrm{V}\,.$$

Obviously, there is

$$\mathrm{I} \supset \mathcal{S}_1, \quad \mathrm{II} \supset \mathcal{S}_2, \quad \mathrm{III} \supset \mathcal{S}_3, \quad \mathrm{IV} \supset \mathcal{R}_0, \quad \mathrm{V} \supset \mathcal{R}_+ \cup \mathcal{R}_-.$$

Hence we get easily

$$\overline{\mathcal{S}_1} \subset \mathcal{S}_1, \quad \overline{\mathcal{S}_2} \subset \mathcal{S}_1 \cup \mathcal{S}_2, \quad \overline{\mathcal{S}_3} \subset \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3,$$
$$\overline{\mathcal{R}_0} \subset \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{R}_0.$$

2.1. **Closure of S_1.** The orbit S_1 consists of a single point, and consequently

$$\overline{S_1} = S_1.$$

For any orbit Ω and any $\omega \in \Omega$ we have $(1/n)\omega \in \Omega$, and

$$\lim_{n\to\infty} (1/n)\omega = 0,$$

which shows that $S_1 \subset \overline{\Omega}$ for any orbit Ω.

2.2. **Closure of $\mathcal{S}_2$.** We have obviously $\overline{\mathcal{S}_2} \supset \mathcal{S}_1 \cup \mathcal{S}_2$. Using the above result we get

$$\overline{\mathcal{S}_2} = \mathcal{S}_1 \cup \mathcal{S}_2.$$

2.3. **Closure of $\mathcal{S}_3$.** Let $\omega \in \mathcal{S}_2$. We can find a basis $\alpha_1, \ldots, \alpha_6$ of $\mathbb{R}^{6*}$ such that $\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$. Next take the form $\theta = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5)$ and use the automorphism $\varphi_n : \mathbb{R}^6 \to \mathbb{R}^6$ such that

$$\varphi_n^* \alpha_i = \alpha_i \text{ for } i \neq 4, \quad \varphi_n^* \alpha_4 = \frac{1}{n}\alpha_4.$$

Setting $\theta_n = \varphi_n^* \theta$ we get $\theta_n \in S_3$ and

$$\theta_n = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \frac{1}{n}\alpha_4 \wedge \alpha_5).$$

Consequently,

$$\lim_{n\to\infty} \theta_n = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \omega.$$

This shows that

$$\overline{\mathcal{S}_3} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3.$$

2.4. **Closure of R_0.** Let $\omega \in S_3$. We can find a basis $\alpha_1, \ldots, \alpha_6$ such that

$$\omega = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5).$$

Consider the form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

and take automorphism φ of $\mathbb{R}^6$ such that

$$\varphi^*\alpha_1 = \alpha_1, \varphi^*\alpha_2 = \alpha_4, \varphi^*\alpha_3 = \alpha_6, \varphi^*\alpha_4 = -\alpha_1 + \alpha_2, \varphi^*\alpha_5 = \alpha_3, \varphi^*\alpha_6 = \alpha_5.$$

Then

$$\theta = \varphi^*\tilde{\theta} = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5) - \alpha_2 \wedge \alpha_4 \wedge \alpha_5 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$$

holds, and applying an automorphism φ_n defined by

$$\varphi_n^*\alpha_1 = \alpha_1, \varphi_n^*\alpha_2 = \frac{1}{n}\alpha_2, \varphi_n^*\alpha_3 = n\alpha_3,$$
$$\varphi_n^*\alpha_4 = \alpha_4, \varphi_n^*\alpha_5 = \alpha_5, \varphi_n^*\alpha_6 = \frac{1}{n^2}\alpha_6.$$

we get

$$\theta_n = \varphi_n^*\theta = \omega - \frac{1}{n}\alpha_2 \wedge \alpha_4 \wedge \alpha_5 + \frac{1}{n}\alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

It is obvious that $\lim_{n\to\infty}\theta_n = \omega$, which shows that $\overline{R_0} \supset S_3$, and that $\overline{R_0} \supset S_1 \cup S_2 \cup S_3 \cup R_0$. Hence we have

$$\overline{R_0} = S_1 \cup S_2 \cup S_3 \cup R_0.$$

2.5. **Closure of R_+.** Now let $\omega \in R_0$. We can find a basis $\alpha_1, \ldots, \alpha_6$ such that

$$\omega = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

Consider the 3-form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$$

and apply to $\tilde{\theta}$ an automorphism φ defined by

$$\begin{aligned} \varphi^*\alpha_1 &= \alpha_1 + \alpha_6, \quad \varphi^*\alpha_2 = \alpha_2 + \alpha_5, \quad \varphi^*\alpha_3 = \alpha_3 + \alpha_4, \\ \varphi^*\alpha_4 &= \alpha_1 - \alpha_6, \quad \varphi^*\alpha_5 = \alpha_2 - \alpha_5, \quad \varphi^*\alpha_6 = \alpha_3 - \alpha_4. \end{aligned}$$

This yields

$$\theta = \varphi^*\tilde{\theta} = 2(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6),$$

where $\theta \in R_+$. Next using the automorphism $\varphi_n : \mathbb{R}^6 \to \mathbb{R}^6$ such that

$$\varphi_n^*\alpha_1 = -\frac{1}{n^2}\alpha_1, \varphi_n^*\alpha_2 = \frac{1}{n^2}\alpha_2, \varphi^*\alpha_3 = -\frac{1}{n^2}\alpha_3, \quad \varphi_n^*\alpha_i = n\alpha_i \text{ for } i = 4, 5, 6.$$

gives

$$\theta_n = \varphi_n^*\theta = 2(\frac{1}{n^6}\alpha_1 \wedge \alpha_1 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6)$$

Consequently,

$$\lim_{n\to\infty} \theta_n = 2\omega,$$

which shows that $\overline{R_+} \supset R_0$. Because R_+ and R_- are disjoint open orbits, we have

$$\overline{R_+} = S_1 \cup S_2 \cup S_3 \cup R_0 \cup R_+.$$

2.6. **Closure of R_-.** We proceed with the same $\omega \in R_0$ as above. This time we take a form

$$\theta = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6,$$

which belongs to R_-. Automorphism φ_n defined as

$$\varphi_n^*\alpha_1 = \frac{1}{n^2}\alpha_1, \varphi_n^*\alpha_2 = \frac{1}{n^2}\alpha_2, \varphi_n^*\alpha_3 = \frac{1}{n^2}\alpha_3, \quad \varphi_n^*\alpha_i = n\alpha_i \text{ for } i = 4, 5, 6,$$

applied to θ gives

$$\theta_n = \varphi_n^*\theta = \frac{1}{n^6}\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6,$$

thus

$$\lim_{n\to\infty} \theta_n = \omega.$$

This shows that $\overline{R_2} \supset R_0$, hence

$$\overline{R_-} = S_1 \cup S_2 \cup S_3 \cup R_0 \cup R_-.$$

3. Dimension 7

Following Djoković we find easily that

$$\overline{\mathrm{I}} = \mathrm{I}, \quad \overline{\mathrm{II}} = \mathrm{I} \cup \mathrm{II}, \quad \overline{\mathrm{III}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}, \quad \overline{\mathrm{IV}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV},$$
$$\overline{\mathrm{V}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V}, \quad \overline{\mathrm{VI}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{VI}, \quad \overline{\mathrm{VII}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{VI} \cup \mathrm{VII},$$
$$\overline{\mathrm{VIII}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI} \cup \mathrm{VII} \cup \mathrm{VIII},$$
$$\overline{\mathrm{IX}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI} \cup \mathrm{VII} \cup \mathrm{VIII} \cup \mathrm{IX},$$
$$\overline{\mathrm{X}} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI} \cup \mathrm{VII} \cup \mathrm{VIII} \cup \mathrm{IX} \cup \mathrm{X}\,.$$

Hence we can see that

$$\overline{\Sigma_1} = \Sigma_1, \quad \overline{\Sigma_2} \subset \Sigma_1 \cup \Sigma_2, \quad \overline{\Sigma_3} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3, \quad \overline{\Omega_0} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0,$$
$$\overline{\Omega_+} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_-, \quad \overline{\Omega_-} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_-,$$
$$(\overline{\Omega_1} \cup \overline{\Omega_6}) \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6,$$
$$(\overline{\Omega_2} \cup \overline{\Omega_7}) \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \Omega_7,$$
$$\overline{\Omega_3} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_3, \quad \overline{\Omega_4} \subset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_3 \cup \Omega_4.$$

3.1. **Closure of Ω_3.** Let us take $\omega \in \Sigma_3$. Then we can find a basis $\alpha_1, \dots, \alpha_6, \alpha_7$ of $(R^7)^*$ such that

$$\omega = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5).$$

Start with the form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5$$

that belongs to Ω_3. Apply an automorphism φ of $\mathbb{R}^7$ such that

$$\varphi^* \alpha_i = \alpha_i \text{ for } i = 1, 2, 4, 5, 6, \quad \varphi^* \alpha_3 = \alpha_7, \varphi^* \alpha_7 = \alpha_3.$$

to get

$$\theta = \varphi^* \tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5.$$

Next apply another automorphism φ_n such that

$$\varphi_n^* \alpha_i = \alpha_i \text{ for } i = 1, 2, 3, 4, 5, 6, \quad \varphi_n^* \alpha_7 = \frac{1}{n} \alpha_7.$$

to see that

$$\theta_n = \varphi_n^* \theta = \omega + \frac{1}{n} \alpha_1 \wedge \alpha_6 \wedge \alpha_7,$$

which shows that $\lim_{n\to\infty} \theta_n = \omega$, and therefore $\overline{\Omega_3} \supset \Sigma_3$. Consequently, we have $\overline{\Omega_3} \supset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_3$. Using the inclusion derived from Djoković, we have

$$\overline{\Omega_3} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_3.$$

3.2. **Closure of Ω_4.** Let $\omega \in \Omega_0$. Then we can find a basis $\alpha_1, \dots, \alpha_6, \alpha_7$ such that

$$\omega = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

We take a form

$$\tilde{\theta} = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_4 \wedge \alpha_6$$

which obviously belongs to Ω_4. Perform an automorphism φ defined by the set of equalities

$$\varphi^*\alpha_1 = \alpha_4,\ \varphi^*\alpha_2 = \alpha_5,\ \varphi^*\alpha_3 = \alpha_7,\ \varphi^*\alpha_4 = \alpha_6,$$
$$\varphi^*\alpha_5 = \alpha_2,\ \varphi^*\alpha_6 = \alpha_3,\ \varphi^*\alpha_7 = \alpha_1$$

on this form to get

$$\theta = \varphi^*\tilde{\theta} = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6 - \alpha_3 \wedge \alpha_4 \wedge \alpha_7$$

Next it suffices to take an automorphism φ_n such that

$$\varphi_n^*\alpha_i = \alpha_i \text{ for } i = 1, \dots, 6 \quad \varphi_n^*\alpha_7 = \frac{1}{n}\alpha_7,$$

to see that

$$\theta_n = \varphi_n^*\theta = \omega - \frac{1}{n}\alpha_3 \wedge \alpha_4 \wedge \alpha_7,$$

and consequently $\lim_{n\to\infty} \theta_n = \omega$. We have thus proved that $\overline{\Omega_4} \supset \Omega_0$, hence

$$\overline{\Omega_4} \supset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_4.$$

Let $\omega \in \Omega_3$. Then we can find a basis $\alpha_1, \dots, \alpha_7$ in $(R^7)^*$ such that

$$\omega = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5).$$

The form

$$\theta = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_4 \wedge \alpha_6$$

obviously belongs to Ω_4, so applying an automorphism φ_n of $\mathbb{R}^7$ such that

$$\varphi_n^*\alpha_i = \alpha_i \text{ for } i \neq 2, 7 \quad \text{and} \quad \varphi_n^*\alpha_2 = \frac{1}{n}\alpha_2, \varphi_n^*\alpha_7 = n\alpha_7.$$

gives

$$\theta_n = \varphi_n^*\theta = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \frac{1}{n}\alpha_2 \wedge \alpha_4 \wedge \alpha_6,$$

which yields $\lim_{n\to\infty} \theta(n) = \omega$. We have thus shown that $\Omega_3 \subset \overline{\Omega_4}$, and consequently that

$$\overline{\Omega_4} \supset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_3 \cup \Omega_4.$$

Now we obviously have

$$\overline{\Omega_4} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_3 \cup \Omega_4.$$

3.3. **Closure of Ω_1.** First, let us consider a converging sequence $\{\omega^{(n)}\}_{n=1}^{\infty}$ of elements from Ω_1 and denote $\omega = \lim_{n\to\infty} \omega^{(n)}$. We introduce an auxiliary scalar product in $\mathbb{R}^7$. Obviously, for every $n \in \mathbb{N}$ there are three orthonormal vectors $v_1^{(n)}, v_2^{(n)}, v_3^{(n)}$ such that

$$(\iota(v_i^{(n)})\omega^{(n)}) \wedge (\iota(v_j^{(n)})\omega^{(n)}) = 0 \quad \text{for } i, j = 1, 2, 3.$$

Considering the triples as elements of the Stiefel manifold $V_{7,3}$, which is compact, we may assume that all the three sequences $\{v_i^{(n)}\}_{n=1}^{\infty}$, $i = 1, 2, 3$ are convergent. We shall denote $v_i = \lim_{n\to\infty} v_i^{(n)}$. Obviously v_1, v_2, v_3 are three orthonormal vectors satisfying

$$(\iota(v_i)\omega) \wedge (\iota(v_j)\omega) = 0, \quad i, j, k = 1, 2, 3,$$

which implies that $[v_1, v_2, v_3] \subset \Delta^2(\omega)$. For $\omega \in \Omega_-$ the subset $\Delta^2(\omega)$ is a 1-dimensional subspace of $\mathbb{R}^7$. Consequently, $\Omega_- \cap \overline{\Omega_1} = \emptyset$.

Now we are going to prove that $\overline{\Omega_1} \supset \Omega_+$. Let $\omega \in \Omega_+$. We can find a basis $\alpha_1, \ldots, \alpha_6, \alpha_7$ such that $\omega = \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6$. All forms from the sequence

$$\omega^{(n)} = \alpha_1 \wedge \alpha_2 \wedge \frac{1}{n}\alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6$$

belong to Ω_1 and $\lim_{n\to\infty} \omega^{(n)} = \omega$. This shows that $\overline{\Omega_1} \supset \Omega_+$.

Next we shall prove that $\overline{\Omega_1} \supset \Omega_4$. If $\omega \in \Omega_4$, then we can find a basis $\alpha_1, \ldots, \alpha_7$ such that

$$\omega = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_4 \wedge \alpha_6.$$

The form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6$$

belongs to Ω_1, hence applying an automorphism φ such that

$$\varphi^*\alpha_1 = \alpha_1,\ \varphi^*\alpha_2 = -\alpha_1 + \alpha_2,\ \varphi^*\alpha_3 = \alpha_3 - \alpha_4,\ \varphi^*\alpha_4 = \alpha_4 - \alpha_5 - \alpha_6,$$
$$\varphi^*\alpha_5 = \alpha_4,\ \varphi^*\alpha_6 = \alpha_6,\ \varphi^*\alpha_7 = \alpha_7.$$

we see that

$$\theta = \varphi^*\tilde{\theta} = \omega + \alpha_1 \wedge \alpha_3 \wedge (\alpha_4 - \alpha_5).$$

Another automorphism φ_n defined as

$$\varphi_n^*\alpha_1 = \alpha_1,\ \varphi_n^*\alpha_2 = \frac{1}{n}\alpha_2,\ \varphi_n^*\alpha_3 = \frac{1}{n}\alpha_3,\ \varphi_n^*\alpha_4 = \alpha_4$$
$$\varphi_n^*\alpha_5 = \alpha_5,\ \varphi_n^*\alpha_6 = n\alpha_6,\ \varphi_n^*\alpha_7 = n\alpha_7.$$

shows that

$$\theta_n = \varphi_n^*\theta = \omega + \frac{1}{n}\alpha_1 \wedge \alpha_3 \wedge (\alpha_4 - \alpha_5),$$

thus $\lim_{n\to\infty} \theta_n = \omega$. This altogether implies that $\overline{\Omega_1} \supset \Omega_4$, and that

$$\overline{\Omega_1} \supset \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_+ \cup \Omega_0 \cup \Omega_1 \cup \Omega_3 \cup \Omega_4.$$

Using the invariant Sign we easily check that $\overline{\Omega_1} \cap \Omega_6 = \emptyset$. Therefore we obtain

$$\overline{\Omega_1} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_+ \cup \Omega_0 \cup \Omega_1 \cup \Omega_3 \cup \Omega_4.$$

3.4. **Closure of Ω_6.** We start with the proof that $\overline{\Omega_6} \supset \Omega_-$. If $\omega \in \Omega_-$, then there exists a basis $\alpha_1, \ldots, \alpha_7$ such that

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

The form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6$$

belongs to Ω_6, so using an automorphism φ defined by

$$\varphi^*\alpha_i = \alpha_i \text{ for } i = 1, 4, 5, 7, \quad \varphi^*\alpha_2 = \alpha_6, \quad \varphi^*\alpha_3 = -\alpha_3, \quad \varphi^*\alpha_6 = -\alpha_2,$$

gets

$$\theta = \varphi^*\tilde{\theta} = \omega + \alpha_1 \wedge \alpha_6 \wedge \alpha_7.$$

Now it suffices to apply an automorphism φ_n such that

$$\varphi_n^*\alpha_i = \alpha_i \text{ for } i \neq 7, \quad \varphi_n^*\alpha_7 = \frac{1}{n}\alpha_7,$$

to see that

$$\theta_n = \varphi_n^*\theta = \omega + \frac{1}{n}\alpha_1 \wedge \alpha_6 \wedge \alpha_7.$$

Then $\lim_{n\to\infty} \theta_n = \omega$ holds, and our assertion $\overline{\Omega_6} \supset \Omega_-$ is proved.

Next we are going to prove that $\overline{\Omega_6} \supset \Omega_4$. Let $\omega \in \Omega_4$. For suitable basis $\alpha_1, \ldots, \alpha_7$ the form ω is equal to

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6.$$

Let us consider the form

$$\theta = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6,$$

which belongs to Ω_6. The automorphism φ_n defined by

$$\varphi_n^* \alpha_1 = \alpha_1, \quad \varphi_n^* \alpha_2 = \frac{1}{n} \alpha_2, \quad \varphi_n^* \alpha_3 = \frac{1}{n} \alpha_3,$$
$$\varphi_n^* \alpha_4 = \alpha_4, \quad \varphi_n^* \alpha_5 = \alpha_5, \quad \varphi_n^* \alpha_6 = n\alpha_6, \quad \varphi_n^* \alpha_7 = n\alpha_7.$$

gives

$$\theta_n = \varphi_n^* \theta = \omega + \frac{1}{n^2} \alpha_2 \wedge \alpha_3 \wedge \alpha_5.$$

Therefore $\lim_{n\to\infty} \theta_n = \omega$, which shows that $\overline{\Omega_6} \supset \Omega_4$, and consequently

$$\overline{\Omega_6} \supset \overline{\Omega_-} \cup \overline{\Omega_4} \cup \Omega_6 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_- \cup \Omega_0 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6.$$

Next we shall prove that $\overline{\Omega_6} \cap \Omega_+ = \emptyset$. The proof will be essentially the same as for Ω_4 type forms (see Subsection 3.2). Let us take again a converging sequence $\{\omega^{(n)}\}_{n=1}^{\infty}$ of elements from Ω_6 such that $\omega = \lim_{n\to\infty} \omega^{(n)} \in \Omega_+$. Obviously $\omega|W \in \mathcal{R}_+$, and by [V], we have that for each $n \in \mathbb{N}$ the forms $\omega^{(n)}|W \in \mathcal{S}_3 \cup \mathcal{R}_- \cup \mathcal{R}_0$ for suitable 6-dimensional W. Consequently, $\omega|W$ belongs to the closure $\overline{\mathcal{S}_3 \cup \mathcal{R}_- \cup \mathcal{R}_0} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{R}_0 \cup \mathcal{R}_-$, a contradiction.

Using the signature of forms, we can easily see that $\Omega_1 \cap \overline{\Omega_6} = \emptyset$. Therefore we obtain

$$\overline{\Omega_6} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_- \cup \Omega_0 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6.$$

3.5. **Closure of Ω_2.** Our first aim is to prove that $\overline{\Omega_2} \supset \Omega_6$. Let $\omega \in \Omega_6$. Then we can find a basis $\alpha_1, \ldots, \alpha_7$ such that

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6.$$

We know that the form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 -$$
$$\alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7.$$

belongs to the orbit Ω_2. We use an automorphism φ defined by

$$\varphi^* \alpha_1 = \alpha_1, \quad \varphi^* \alpha_2 = -\alpha_2, \quad \varphi^* \alpha_3 = \alpha_2 + \alpha_3, \quad \varphi^* \alpha_4 = \alpha_1 + \alpha_4,$$
$$\varphi^* \alpha_5 = -\alpha_5 - \alpha_6 - \alpha_7, \quad \varphi^* \alpha_6 = -\alpha_5 + \alpha_6, \quad \varphi^* \alpha_7 = \alpha_5.$$

We get

$$\theta = \varphi^* \tilde{\theta} = \omega + \alpha_3 \wedge \alpha_4 \wedge \alpha_6.$$

Next step consists in applying an automorphism φ_n^* defined as follows.

$$\varphi_n^* \alpha_1 = n\alpha_1, \quad \varphi_n^* \alpha_2 = n\alpha_2, \quad \varphi_n^* \alpha_3 = \alpha_3, \quad \varphi_n^* \alpha_4 = \alpha_4,$$
$$\varphi_n^* \alpha_5 = \frac{1}{n} \alpha_5, \quad \varphi_n^* \alpha_6 = \frac{1}{n} \alpha_6, \quad \varphi_n^* \alpha_7 = \frac{1}{n^2} \alpha_7.$$

We set $\theta_n = \varphi_n^* \theta$. It is obvious that $\theta_n \in \Omega_2$ for every $n \in \mathbb{N}$. Furthermore,

$$\theta_n = \omega + \frac{1}{n} \alpha_3 \wedge \alpha_4 \wedge \alpha_6.$$

This proves $\lim_{n\to\infty}\theta_n = \omega$, hence $\overline{\Omega_2} \supset \Omega_6$.

The next step is to show that $\overline{\Omega_2} \supset \Omega_1$. For $\omega \in \Omega_1$ we find a basis $\alpha_1, \dots, \alpha_7$ such that

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6.$$

The form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 -$$

$$\alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7$$

belongs to the orbit Ω_2, hence an automorphism φ defined by

$$\varphi^*\alpha_1 = \alpha_1, \quad \varphi^*\alpha_2 = \alpha_2, \quad \varphi^*\alpha_3 = -\alpha_5, \quad \varphi^*\alpha_4 = \alpha_2 + \alpha_4,$$
$$\varphi^*\alpha_5 = 2\alpha_3 + \alpha_7, \quad \varphi^*\alpha_6 = \alpha_3 + \alpha_6, \quad \varphi^*\alpha_7 = -\alpha_3.$$

shows that

$$\theta = \varphi^*\tilde{\theta} = \omega + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6.$$

What we need to do is to apply an automorphism φ_n defined as below.

$$\varphi_n^*\alpha_1 = n^2\alpha_1, \quad \varphi_n^*\alpha_2 = \alpha_2, \quad \varphi_n^*\alpha_3 = \frac{1}{n}\alpha_3, \quad \varphi_n^*\alpha_4 = \frac{1}{n}\alpha_4,$$
$$\varphi_n^*\alpha_5 = \alpha_5, \quad \varphi_n^*\alpha_6 = \alpha_6, \quad \varphi_n^*\alpha_7 = \frac{1}{n^2}\alpha_7.$$

If we set $\theta_n = \varphi_n^*\theta$, we see that $\theta_n \in \Omega_2$ and $\lim_{n\to\infty}\theta_n = \omega$, thus $\overline{\Omega_2} \supset \Omega_1$. Summarizing, we have

$$\overline{\Omega_2} \supset \overline{\Omega_1} \cup \overline{\Omega_6} \cup \Omega_2 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6.$$

Using [D2] we find that $\overline{\Omega_2} \cap \Omega_5 = \emptyset$ and $\overline{\Omega_2} \cap \Omega_8 = \emptyset$. Using the invariant *Sign* we find that $\overline{\Omega_2} \cap \Omega_7 = \emptyset$. Consequently we obtain

$$\overline{\Omega_2} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6.$$

3.6. **Closure of Ω_7.** We shall prove first that $\overline{\Omega_7} \supset \Omega_6$. Let $\omega \in \Omega_6$. Then we can find a basis $\alpha_1, \dots, \alpha_7$ such that

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6.$$

Let us take an element

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 +$$

$$\alpha_2 \wedge \alpha_3 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5,$$

which obviously belongs to Ω_7. We use an automorphism φ defined by

$$\varphi^*\alpha_i = \alpha_i \text{ for } i = 1, 2, 3, 4, \quad \varphi^*\alpha_5 = \alpha_7, \quad \varphi^*\alpha_6 = -\alpha_6, \quad \varphi^*\alpha_7 = \alpha_5.$$

We get $\theta = \varphi^*\tilde{\theta} = \omega + \alpha_3 \wedge \alpha_4 \wedge \alpha_7$. Next we perform an automorphism φ_n defined by

$$\varphi_n^*\alpha_1 = \alpha_1, \quad \varphi_n^*\alpha_2 = \alpha_2, \quad \varphi_n^*\alpha_3 = \frac{1}{n}\alpha_3, \quad \varphi_n^*\alpha_4 = \frac{1}{n}\alpha_4,$$
$$\varphi_n^*\alpha_5 = n\alpha_5, \quad \varphi_n^*\alpha_6 = n\alpha_6, \quad \varphi_n^*\alpha_7 = \alpha_7.$$

We have

$$\theta_n = \varphi_n^*\theta = \omega + \frac{1}{n^2}\alpha_3 \wedge \alpha_4 \wedge \alpha_7,$$

and consequently $\lim_{n\to\infty}\theta_n = \omega$. We have proved that $\overline{\Omega_7} \supset \Omega_6$. In this way we get

$$\overline{\Omega_7} \supset \overline{\Omega_6} \cup \Omega_7 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_- \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \Omega_7.$$

Next we can easily verify that $\overline{\Omega_7} \cap \Omega_i = \emptyset$ for $i = 1, 2$ (invariant $Sign$). The results of [D2] show that $\overline{\Omega_7} \cap \Omega_5 = \emptyset$ and $\overline{\Omega_7} \cap \Omega_8 = \emptyset$. Plus, a similar argument as in Subsections 3.2, 3.4 again shows that $\overline{\Omega_7} \cap \Omega_+ = \emptyset$. This time $\omega^{(n)}|W \in \mathcal{R}_- \cup \mathcal{R}_0$, hence $\omega|W \in \overline{\mathcal{R}_- \cup \mathcal{R}_0} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{R}_0 \cup \mathcal{R}_-$, and our claim follows.

Consequently, we have

$$\overline{\Omega_7} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_- \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \Omega_7.$$

3.7. **Closure of Ω_5.** We are going to prove first that $\Omega_2 \subset \overline{\Omega_5}$. To do so, let $\omega \in \Omega_2$. Then we can find a basis $\alpha_1, \ldots, \alpha_7$ such that

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 - \alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6$$

Let us take an element

$$\begin{aligned}\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 \\ + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.\end{aligned}$$

This form obviously belongs to Ω_5. We perform now an automorphism φ defined by

$$\begin{aligned}\varphi^*\alpha_1 = \alpha_1,\ \varphi^*\alpha_2 = -\alpha_2,\ \varphi^*\alpha_3 = -\alpha_5,\ \varphi^*\alpha_4 = \alpha_6, \\ \varphi^*\alpha_5 = \alpha_3,\ \varphi^*\alpha_6 = \alpha_4,\ \varphi^*\alpha_7 = \alpha_7.\end{aligned}$$

$$\begin{aligned}\varphi^*\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 - \alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_6 \\ + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_5 \wedge \alpha_6 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5\end{aligned}$$

We define now a new form

$$\theta = \varphi^*\tilde{\theta} + 2a\alpha_3 \wedge \alpha_4 \wedge \alpha_6,$$

and we are going to prove that for any $a \in \mathbb{R}$ the form θ belongs to Ω_5. For simplicity we denote $\mu = \varphi^*\tilde{\theta}$ and $\nu = \alpha_3 \wedge \alpha_4 \wedge \alpha_6$. For any $v \in V$ we have

$$(\iota_v\theta) \wedge (\iota_v\theta) \wedge \theta = (\iota_v\mu) \wedge (\iota_v\mu) \wedge \mu + 2a(\iota_v\mu) \wedge (\iota_v\mu) \wedge \nu + 4a(\iota_v\mu) \wedge (\iota_v\nu) \wedge \mu.$$

Let $e_1, \ldots, e_7$ be a dual basis to $\alpha_1, \ldots, \alpha_7$ and let us write $v = c_1e_1 + \cdots + c_7e_7$. We find then

$$\begin{aligned}(\iota_v\mu) \wedge (\iota_v\mu) \wedge \mu &= 6(-c_1^2 - c_2^2 + c_3^2 + c_4^2 - c_5^2 + c_6^2 + c_7^2)\alpha_1 \wedge \cdots \wedge \alpha_7, \\ (\iota_v\mu) \wedge (\iota_v\mu) \wedge \nu &= 2(c_1c_4 - c_2c_3 - c_5c_6)\alpha_1 \wedge \cdots \wedge \alpha_7, \\ (\iota_v\mu) \wedge (\iota_v\nu) \wedge \mu &= 4(c_1c_4 - c_2c_3 - c_5c_6)\alpha_1 \wedge \cdots \wedge \alpha_7.\end{aligned}$$

Consequently, we obtain

$$\frac{1}{6}(\iota_v\theta) \wedge (\iota_v\theta) \wedge \theta =$$
$$(-c_1^2 - c_2^2 + c_3^2 + c_4^2 - c_5^2 + c_6^2 + c_7^2 + 2ac_1c_4 - 2ac_2c_3 - 2ac_5c_6)\alpha_1 \wedge \cdots \wedge \alpha_7.$$

Computing the determinant D of the quadratic form on the right hand side, we find that $D = -(1+a^2)^3$. This shows that for any $a \in \mathbb{R}$ the form θ belongs to the orbit Ω_5.

We set now $a = n$ and apply an automorphism ψ_n given by

$$\psi^*\alpha_i = \alpha_i \text{ for } i \neq 6, \quad \psi^*\alpha_6 = \frac{1}{2n}\alpha_6 - \frac{1}{2n}\alpha_5.$$

$$\begin{aligned}\theta_n = \psi_n^*\theta = \alpha_1\wedge\alpha_2\wedge\alpha_5 + \alpha_1\wedge\alpha_4\wedge\alpha_7 - \alpha_2\wedge\alpha_3\wedge\alpha_7 + \alpha_3\wedge\alpha_4\wedge\alpha_6 \\ -\frac{1}{2n}\alpha_1\wedge\alpha_3\wedge\alpha_5 + \frac{1}{2n}\alpha_1\wedge\alpha_3\wedge\alpha_6 - \frac{1}{2n}\alpha_2\wedge\alpha_4\wedge\alpha_5 \\ +\frac{1}{2n}\alpha_2\wedge\alpha_4\wedge\alpha_6 - \frac{1}{2n}\alpha_5\wedge\alpha_6\wedge\alpha_7\end{aligned}$$

Obviously $\theta_n \in \Omega_5$ and we have $\lim_{n\to\infty}\theta_n = \omega$. We have thus proved that $\Omega_2 \subset \overline{\Omega_5}$. We have thus shown that

$$\overline{\Omega_5} \supset \overline{\Omega_2} \cup \Omega_5 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5 \cup \Omega_6.$$

Next we are going to prove that $\overline{\Omega_5} \supset \Omega_7$. Let $\omega \in \Omega_7$. Then we can find a basis $\alpha_1, \dots, \alpha_7$ such that

$$\begin{aligned}\omega = \alpha_1\wedge\alpha_2\wedge\alpha_5 + \alpha_1\wedge\alpha_3\wedge\alpha_6 + \alpha_1\wedge\alpha_4\wedge\alpha_7 \\ +\alpha_2\wedge\alpha_3\wedge\alpha_7 - \alpha_2\wedge\alpha_4\wedge\alpha_6 + \alpha_3\wedge\alpha_4\wedge\alpha_5.\end{aligned}$$

We know that the form

$$\begin{aligned}\tilde\theta = \alpha_1\wedge\alpha_2\wedge\alpha_3 - \alpha_1\wedge\alpha_4\wedge\alpha_5 + \alpha_1\wedge\alpha_6\wedge\alpha_7 + \alpha_2\wedge\alpha_4\wedge\alpha_6 \\ +\alpha_2\wedge\alpha_5\wedge\alpha_7 + \alpha_3\wedge\alpha_4\wedge\alpha_7 - \alpha_3\wedge\alpha_5\wedge\alpha_6\end{aligned}$$

belongs to Ω_5. We perform an automorphism φ such that

$$\begin{aligned}\varphi^*\alpha_1 = \alpha_7, \varphi^*\alpha_2 = \alpha_6, \varphi^*\alpha_3 = -\alpha_5, \varphi^*\alpha_4 = \alpha_4, \\ \varphi^*\alpha_5 = \alpha_1, \varphi^*\alpha_6 = \alpha_2, \varphi^*\alpha_7 = \alpha_3.\end{aligned}$$

We get

$$\begin{aligned}\theta = \varphi^*\tilde\theta = \alpha_5\wedge\alpha_6\wedge\alpha_7 + \alpha_1\wedge\alpha_4\wedge\alpha_7 + \alpha_2\wedge\alpha_3\wedge\alpha_7 - \alpha_2\wedge\alpha_4\wedge\alpha_6 \\ +\alpha_1\wedge\alpha_3\wedge\alpha_6 + \alpha_3\wedge\alpha_4\wedge\alpha_5 + \alpha_1\wedge\alpha_2\wedge\alpha_5 = \\ = \alpha_5\wedge\alpha_6\wedge\alpha_7 + \omega.\end{aligned}$$

We define now an automorphism φ_n by the formulas

$$\begin{aligned}\varphi_n^*\alpha_1 = n\alpha_1, \varphi_n^*\alpha_2 = n\alpha_2, \varphi_n^*\alpha_3 = n\alpha_3, \varphi_n^*\alpha_4 = n\alpha_4, \\ \varphi_n^*\alpha_5 = \frac{1}{n^2}\alpha_5, \varphi_n^*\alpha_6 = \frac{1}{n^2}\alpha_6, \varphi_n^*\alpha_7 = \frac{1}{n^2}\alpha_7.\end{aligned}$$

Then we get

$$\theta_n = \varphi_n^*\theta = \omega + \frac{1}{n^6}\alpha_5\wedge\alpha_6\wedge\alpha_7,$$

which shows that $\lim_{n\to\infty}\theta_n = \omega$. We have thus shown that $\overline{\Omega_5} \supset \Omega_7$. Using the "'Sign" invariant we find $\overline{\Omega_5} \cap \Omega_8 = \emptyset$, and we obtain

$$\overline{\Omega_5} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5 \cup \Omega_6 \cup \Omega_7.$$

3.8. **Closure of Ω_8.** We shall prove that $\overline{\Omega_8} \supset \Omega_7$. We can find a basis $\alpha_1, \ldots, \alpha_7$ of $\mathbb{R}^{7*}$ such that

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7$$

$$+\alpha_2 \wedge \alpha_3 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5.$$

Then we take a form

$$\tilde{\theta} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7$$

$$+\alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7$$

$$-\alpha_3 \wedge \alpha_5 \wedge \alpha_6$$

which belongs to the orbit Ω_8. Now we use an automorphism φ defined by

$$\varphi^*\alpha_1 = -\alpha_5, \varphi^*\alpha_2 = \alpha_2, \varphi^*\alpha_3 = \alpha_1, \varphi^*\alpha_4 = \alpha_4,$$

$$\varphi^*\alpha_5 = alpha_3, \varphi^*\alpha_6 = -alpha_6, \varphi^*\alpha_7 = alpha_7.$$

We obtain

$$\theta = \varphi^*\tilde{\theta} = \omega - \alpha_5 \wedge \alpha_6 \wedge \alpha_7.$$

Next using the automorphism φ_n defined by

$$\varphi^*\alpha_i = n\alpha_i \text{ for } i = 1,2,3,4, \quad \varphi^*\alpha_j = \frac{1}{n^2}\alpha_j \text{ for } j = 5,6,7$$

we have

$$\theta_n = \varphi_n^*\theta = \omega - \frac{1}{n^6}\alpha_5 \wedge \alpha_6 \wedge \alpha_7.$$

Consequently we have $\lim_{n\to\infty}\theta_n = \omega$, which shows that $\overline{\Omega_8} \supset \Omega_7$. We have therefore

$$\overline{\Omega_8} \supset \overline{\Omega_7} \cup \Omega_8 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_- \cup \Omega_0 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \Omega_7 \cup \Omega_8.$$

It is not difficult to see that $\overline{\Omega_8} \cap \Omega_+ = \emptyset$. Let us assume that $\omega \in \Omega_+$ belongs to $\overline{\Omega_8}$. Then there exists a sequence $\{\omega^{(n)}\}_{n=1}^{\infty}$ of elements from Ω_8 such that $\lim_{n\to\infty}\omega^{(n)} = \omega$. Obviously we can find a 6-dimensional subspace $W \subset \mathbb{R}^7$ such that $\omega|W$ belongs to the orbit $\mathcal{R}_+$. According to [V] $\omega^{(n)}|W$ belongs to the orbit $\mathcal{R}_-$ for every n. We can see that $\lim_{n\to\infty}\omega^{(n)}|W = \omega|W$, which shows that $\overline{\mathcal{R}_-} \cap \mathcal{R}_+ \neq \emptyset$, and this is a contradiction. Using the invariant *Sign* we find that $\overline{\Omega_8} \cap \Omega_i = \emptyset$ for $i = 1,2,5$. Then we obtain

$$\overline{\Omega_8} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Omega_- \cup \Omega_0 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6 \cup \Omega_7 \cup \Omega_8.$$

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